

# Vector Field Curvature and Applications

## Dissertation

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## Abstract

The treatment of tangent curves is a powerful tool for analyzing and visualizing the behavior of vector fields. Unfortunately, for sufficiently complicated vector fields, the tangent curves can only be implicitly described as the solution of a system of differential equations.

In this work we show how to compute the curvature of tangent curves and discuss its usefulness for analyzing and visualizing vector fields. In particular, we investigate the curvature behavior around critical points. We show that the curvature of the tangent curves of a 2D vector field and its perpendicular vector field uniquely describe the vector directions in the vector field. We will also describe special curvature properties of linear vector fields in 2D.

Applying the ideas of vector field curvature to vector fields over general parametrized surfaces, we are able to compute the curvature of particular tangent curves on a surface, such as contour lines, lines of curvature, asymptotic lines, isophotes and reflection lines. For special tangent curves, we introduce "thickness" as another characteristic measure. We discuss the application of the curvature of tangent curves on surfaces as a surface interrogation tool.

Finally, using the concepts of curvature of tangent curves, we deduce geometric conditions (necessary and sufficient) for  $G^3$  continuity of surfaces.

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# Chapter 1

## Introduction

The visualization of vector fields has become one of the main topics in scientific visualization: CFD-data is usually given in form of vector fields. One of the most powerful tools for analyzing and visualizing vector fields is the treatment of their tangent curves. Unfortunately, for sufficiently complicated vector fields, those curves can only be described implicitly as the solution of a system of differential equations.

Although we don't have an explicit formula for the tangent curves, we can compute their curvatures in an easy way if we know the vector field and its partial derivatives. This is the main idea of chapter 2. Also in chapter 2, we introduce the concepts of perpendicular and rotated vector fields in 2D and discuss the curvature behavior of their tangent curves.

In section 2.5 we explore the curvature behavior around critical points. In theorem 1 we will show that the curvature near critical points tends to infinity – at least for one direction in the vector field or in the perpendicular vector field. This property will be useful for detecting critical points in the curvature visualization of vector fields.

In section 2.6 we show that the direction of the vectors in a vector field are uniquely described by the curvature of the tangent curves of the original and its perpendicular vector field.

Section 2.7 treats the special case of linear vector fields. For those vector fields some more characteristic curvature properties apply.

Chapter 2 ends with a first approach for extending the concept of vector field curvature to 3D vector fields.

In chapter 3 we discuss the usage of tangent curve curvature as a visualization technique for vector fields. An assessment and examples for the technique are given.

In chapter 4 we lay the foundations for expanding the concept of vector field curvature to tangent curves on general parametrized surfaces. For special vector fields we introduce the "thickness" of the tangent curves as another characteristic property.

In chapter 5 the theoretical results of chapter 4 are applied to particular tangent curves on surfaces. These curves are contour lines, lines of curvature, asymptotic lines, isophotes and reflection lines. We show how to compute their curvature, their geodesic curvature, and (if possible) their "thickness". Furthermore, we investigate the conditions for the appearance of critical points of those curve families. Finally, we discover geometric conditions (necessary and sufficient) for  $G^3$  continuity of surfaces. These conditions are based on the  $G^2$ -continuity of lines of curvature and asymptotic lines, and are formulated in the theorems 5 and 6.

Chapter 6 discusses the application of the curvature of tangent curves on surfaces as a surface interrogation tool. Examples for a test surface are given.

In chapter 7, open questions for future research are formulated.

The color pictures of the chapters 3 and 6 can also be found at <http://enuxsa.eas.asu.edu:80/~theisel/> .



# Chapter 2

## The Theory of Vector Field Curvature

This chapter gives the theoretical background of the entire work. The concept of vector field curvature is introduced for 2D vector fields, and fundamental properties are proven. The chapter ends with the treatment of the special case of linear vector fields and an extension of some properties to the 3D case.

### 2.1 Basic Definitions and Notations

Let  $\mathbb{E}^2$  be the euclidian plane, equipped with a cartesian coordinate system. This way a point  $P \in \mathbb{E}^2$  can be described by two real numbers  $u, v$  (Notation:  $P \sim (u, v)$ ). Let  $\mathbb{R}^2$  be the associated 2-dimensional vector space. A map

$$V : \mathbb{E}^2 \rightarrow \mathbb{R}^2 \tag{2.1}$$

is called *vector field*.  $V$  assigns a vector  $(vx(P), vy(P))^T$  to any point  $P \sim (u, v)$ . We use the notation  $V(P) = V(u, v) = (vx(u, v), vy(u, v))^T$ .

The *partial derivatives* of  $V$  are defined as  $V_u(u, v) = (vx_u(u, v), vy_u(u, v))^T$ , similar for  $V_v$  and higher order partial derivatives. For convenience we assume that the vector field is piecewise analytic, i.e. all partial derivatives of  $V$  are defined and continuous.

A point  $P \in \mathbb{E}^2$  is called *critical point of  $V$*  if  $V(P) = \mathbf{0}$  is the zero vector.

Given a vector field  $V$ , we can define its *normalized* (or *direction*) *vector field*  $\bar{V} = (\bar{v}_x, \bar{v}_y)^T$ :

$$\bar{V} := \frac{V}{\|V\|}. \quad (2.2)$$

(This definition works only for non-critical points. For critical points,  $\bar{V}$  is not defined.) For a normalized vector field we have  $\bar{V} \cdot \bar{V} = 1$  (where " $\cdot$ " denotes the usual dot product of vectors). Differentiating this equation in  $u$ - and  $v$ -direction yields

$$\bar{V} \cdot \bar{V}_u = 0 \quad (2.3)$$

$$\bar{V} \cdot \bar{V}_v = 0. \quad (2.4)$$

Since we are in 2D, (2.3) and (2.4) give

$$\det[\bar{V}_u, \bar{V}_v] = 0. \quad (2.5)$$

The partial derivatives of  $\bar{V}$  are perpendicular to  $\bar{V}$  and linearly dependent.

Now we want to introduce the concept of tangent curves of a vector field.

**Definition 1** *A curve  $L \subseteq \mathbb{E}^2$  is called tangent curve (stream line, flow line, characteristic curve) of the vector field  $V$  if the following condition is satisfied: For all points  $P \in L$ , the tangent vector of the curve in the point  $P$  has the same direction as the vector  $V(P)$ .*

Figure 2.1 gives an illustration of this definition.

For every point  $P \in \mathbb{E}^2$  there is one and only one tangent curve through it (except for critical points of  $V$ ). Tangent curves do not intersect each other (except for critical points of  $V$ ). They do not depend on the magnitudes of the vectors in the vector field: two vector fields which have vectors of the same direction (but not necessary of the same magnitude) in every point produce identical tangent curves. (For instance, the tangent curves produced by a vector field  $V$  and its normalized vector field  $\bar{V}$  are identical.)

Tangent curves play an important role for both analysis and visualization of vector fields. If the vector field describes the flow of a fluid or gas, the

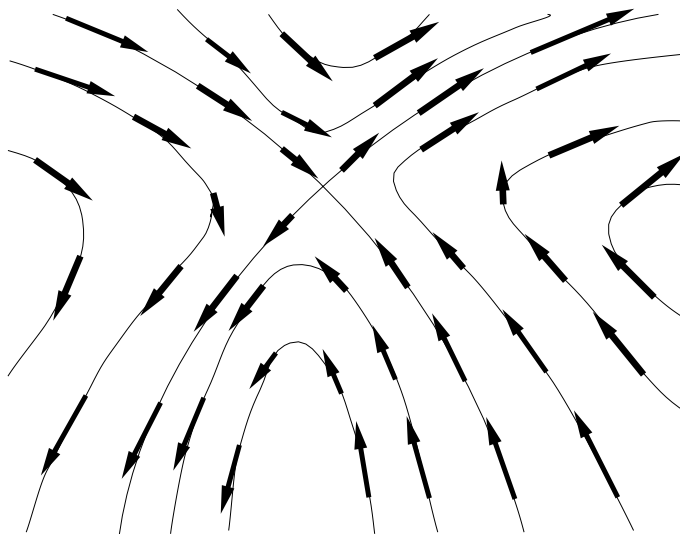


Figure 2.1: Vector field and tangent curves around a critical point

tangent curves can be considered as the path of a massless particle in the flow.

A tangent curve  $L(t) = (u(t), v(t))^T$  of the vector field  $V = (vx, vy)^T$  can be described as the solution of the system of differential equations

$$\frac{du}{dt} = vx(u, v) \tag{2.6}$$

$$\frac{dv}{dt} = vy(u, v). \tag{2.7}$$

Unfortunately, for sufficiently complicated vector fields there is no closed solution of (2.6) and (2.7): tangent curves are in general not describable as parametrized curves but only in the implicit form of (2.6) and (2.7).

## 2.2 Tangent Curves and Curvature

Although we don't know the tangent curves in a parametric description we want to ask for geometric properties of these curves. One of the most important geometric properties is their (signed) curvature. In order to compute the curvature of a tangent curve, we need the following

**Lemma 1** Let  $V = (vx, vy)^T$  be a vector field, let  $L(t)$  be an arbitrary tangent curve in  $V$ , and let  $P \sim (u, v)$  be an arbitrary point on  $L$ . Furthermore, let  $L(t)$  be parametrized in a way that  $P = L(t_0)$  and  $\dot{L}(t) = V(L(t))$  for every  $t$ . ( $\dot{L}(t)$  denotes the tangent vector of  $L$  at  $t$ .)  
Then we obtain for the second derivative vector of  $L$ :

$$\ddot{L}(t_0) = (vx \cdot V_u + vy \cdot V_v)(P). \quad (2.8)$$

**Proof:** Applying the chain rule to  $\dot{L}(t) = V(L(t))$  gives

$$\dot{V} = \frac{dV}{dt} = V_u \cdot \frac{du}{dt} + V_v \cdot \frac{dv}{dt} = vx \cdot V_u + vy \cdot V_v = \ddot{L}. \quad \square \quad (2.9)$$

Lemma 1 has the following consequence: in order to get the first and the second derivative vector of a tangent curve in a point  $P$ , it is not necessary to know the tangent curve itself. It is sufficient to know the vector field  $V$  and its partial derivatives.

If we know the first and the second derivative vector  $\dot{L}(t_0)$  and  $\ddot{L}(t_0)$  of the tangent curve in the point  $P = L(t_0)$ , we can easily compute the signed curvature of  $L$  in  $P$ :

$$\kappa(t_0) = \frac{\det [\dot{L}(t_0), \ddot{L}(t_0)]}{\|\dot{L}(t_0)\|^3}. \quad (2.10)$$

This and (2.8) gives

$$\kappa(t_0) = \frac{(vx^2 \cdot vy_u - vy^2 \cdot vx_v + vx \cdot vy \cdot (vy_v - vx_u))(P)}{\|V(P)\|^3}. \quad (2.11)$$

We have computed the curvature of a particular point on a particular tangent curve. Since we know that there is one and only one tangent curve through every point of the vector field (except for critical points), it makes sense to give the following

**Definition 2** The curvature  $\kappa$  of a vector field  $V(u, v)$  is a scalar field over the  $(u, v)$ -domain which contains the curvature of the tangent curve through  $(u, v)$  in the point  $(u, v)$  for every point  $(u, v)$  of the domain.

Again we have to mention that  $\kappa(V)$  is defined only for non-critical points in  $V$ . From (2.11) we can easily deduce the formula of  $\kappa(V)$ :

$$\kappa(V) = \frac{vx^2 \cdot vy_u - vy^2 \cdot vx_v + vx \cdot vy \cdot (vy_v - vx_u)}{\|V\|^3}. \quad (2.12)$$

Since  $V$  is piecewise analytic and  $\kappa$  is a scalar field, the partial derivatives  $\kappa_u$  and  $\kappa_v$  of  $\kappa$  (which are scalar fields as well) exist and can be derived by applying basic differentiation rules to (2.12).

**Remark 1:** Since the tangent curves do not depend on the magnitudes of the vectors in  $V$ ,  $\kappa(V)$  does not depend on the magnitudes of the vectors in  $V$  as well.

## 2.3 Curvature of Rotated Vector Fields

In this section we introduce the concepts of rotated and perpendicular vector fields and investigate their curvature. The results form a foundation of some properties of vector fields described later.

Given a vector field  $V$  and an angle  $\gamma$ , we can produce a new vector field  $V^{[\gamma]}$ : for every point  $P$  the direction of  $V(P)$  is rotated counterclockwise by  $\gamma$ , and the magnitude remains unchanged:

$$V^{[\gamma]} = \begin{pmatrix} vx^{[\gamma]} \\ vy^{[\gamma]} \end{pmatrix} = \begin{pmatrix} vx \cdot \cos \gamma - vy \cdot \sin \gamma \\ vx \cdot \sin \gamma + vy \cdot \cos \gamma \end{pmatrix}. \quad (2.13)$$

$V^{[\gamma]}$  is called the *rotated vector field* of  $V$  by the angle  $\gamma$ . (See figure 2.2 for an illustration. )

Let  $\kappa^{[\gamma]}(V)$  be the curvature of the rotated vector field  $V^{[\gamma]}$ :

$$\kappa^{[\gamma]}(V) := \kappa(V^{[\gamma]}). \quad (2.14)$$

$\kappa^{[\gamma]}(V)$  is called the *rotated curvature* of  $V$  by the angle  $\gamma$ .

We consider the special cases  $\gamma = 0$  and  $\gamma = \frac{\pi}{2}$ . For  $\gamma = 0$  we can compute  $\kappa^{[0]}(P) = \kappa(P)$  by (2.12). We want to call  $V^{[\frac{\pi}{2}]}$  the *perpendicular vector field*

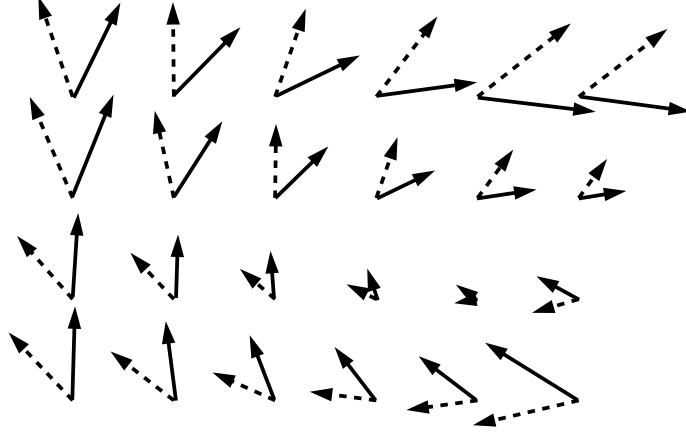


Figure 2.2: If the solid arrows denote the vector field  $V^{[0]}$ , the dashed arrows denote the vector field  $V^{[\frac{\pi}{4}]}$ .

of  $V$  and  $\kappa^{[\frac{\pi}{2}]}$  the *perpendicular curvature* of  $V$ . We obtain from (2.13):

$$V^{[\frac{\pi}{2}]} = \begin{pmatrix} vx^{[\frac{\pi}{2}]} \\ vy^{[\frac{\pi}{2}]} \end{pmatrix} = \begin{pmatrix} -vy \\ vx \end{pmatrix} \quad (2.15)$$

$$V^{[\frac{\pi}{2}]_u} = \begin{pmatrix} vx^{[\frac{\pi}{2}]_u} \\ vy^{[\frac{\pi}{2}]_u} \end{pmatrix} = \begin{pmatrix} -vy_u \\ vx_u \end{pmatrix} \quad (2.16)$$

$$V^{[\frac{\pi}{2}]_v} = \begin{pmatrix} vx^{[\frac{\pi}{2}]_v} \\ vy^{[\frac{\pi}{2}]_v} \end{pmatrix} = \begin{pmatrix} -vy_v \\ vx_v \end{pmatrix}. \quad (2.17)$$

From (2.15),(2.16), (2.17) and (2.12) we obtain

$$\kappa^{[\frac{\pi}{2}]} = \frac{vx^2 \cdot vy_v + vy^2 \cdot vx_u - vx \cdot vy \cdot (vx_v + vy_u)}{\|V\|^3}. \quad (2.18)$$

For an arbitrary angle  $\gamma$  we obtain from (2.13):

$$V^{[\gamma]_u}(P) = \begin{pmatrix} vx_u \cdot \cos \gamma - vy_u \cdot \sin \gamma \\ vx_u \cdot \sin \gamma + vy_u \cdot \cos \gamma \end{pmatrix} \quad (2.19)$$

$$V^{[\gamma]_v}(P) = \begin{pmatrix} vx_v \cdot \cos \gamma - vy_v \cdot \sin \gamma \\ vx_v \cdot \sin \gamma + vy_v \cdot \cos \gamma \end{pmatrix}. \quad (2.20)$$

(2.12),(2.18), (2.13),(2.19) and (2.20) yield

$$\kappa^{[\gamma]} = \kappa^{[0]} \cdot \cos \gamma + \kappa^{[\frac{\pi}{2}]} \cdot \sin \gamma. \quad (2.21)$$

Given the curvature and the perpendicular curvature of a vector field, we can compute the rotated curvature  $\kappa^{[\gamma]}$  for every angle  $\gamma$  using the simple formula (2.21).

Now we want to find the extreme values of  $\kappa^{[\gamma]}$  for all angles  $\gamma$ . This will be useful for the proofs in the next sections.

Let  $\kappa_{\max} := \max \{ \kappa^{[\gamma]} : \gamma \in \langle 0, 2\pi \rangle \}$  and  $\kappa_{\min} := \min \{ \kappa^{[\gamma]} : \gamma \in \langle 0, 2\pi \rangle \}$ . From (2.21) it can be shown that

$$\kappa_{\max} = \sqrt{(\kappa^{[0]})^2 + (\kappa^{[\frac{\pi}{2}]})^2}. \quad (2.22)$$

(2.12), (2.18) and (2.22) yield

$$\kappa_{\max} = \frac{\sqrt{(vx \cdot vy_u - vy \cdot vx_u)^2 + (vx \cdot vy_v - vy \cdot vx_v)^2}}{\|V\|^2}. \quad (2.23)$$

Since  $\kappa^{[\gamma+\pi]} = -\kappa^{[\gamma]}$ , we obtain

$$\kappa_{\min} = -\kappa_{\max}. \quad (2.24)$$

## 2.4 Alternative Notations of Vector Field Curvature

After showing how to compute the curvature  $\kappa$  of a vector field, we want to introduce some alternative (and sometimes easier) expressions of  $\kappa$ .

We can describe the vectors of a vector field not only in terms of  $vx$ - and  $vy$ -components but also in polar form: as an angle  $\phi$  and a magnitude  $m$ :

$$V = \begin{pmatrix} vx \\ vy \end{pmatrix} = \begin{pmatrix} m \cdot \cos \phi \\ m \cdot \sin \phi \end{pmatrix}. \quad (2.25)$$

This gives for the partial derivatives:

$$V_u = \begin{pmatrix} vx_u \\ vy_u \end{pmatrix} = \begin{pmatrix} m_u \cdot \cos \phi - m \cdot \phi_u \cdot \sin \phi \\ m_u \cdot \sin \phi + m \cdot \phi_u \cdot \cos \phi \end{pmatrix} \quad (2.26)$$

$$V_v = \begin{pmatrix} vx_v \\ vy_v \end{pmatrix} = \begin{pmatrix} m_v \cdot \cos \phi - m \cdot \phi_v \cdot \sin \phi \\ m_v \cdot \sin \phi + m \cdot \phi_v \cdot \cos \phi \end{pmatrix}. \quad (2.27)$$

Applying (2.25), (2.26) and (2.27) to (2.12) and (2.18), we obtain:

$$\kappa = \kappa^{[0]} = \phi_u \cdot \cos \phi + \phi_v \cdot \sin \phi \quad (2.28)$$

$$\kappa^{\left[\frac{\pi}{2}\right]} = -\phi_u \cdot \sin \phi + \phi_v \cdot \cos \phi \quad (2.29)$$

$$\kappa_{\max} = \sqrt{\phi_u^2 + \phi_v^2}. \quad (2.30)$$

Another possibility of expressing  $\kappa$  is the usage of the normalized vector field  $\bar{V}$ .

$$\bar{V} = \begin{pmatrix} \bar{v}x \\ \bar{v}y \end{pmatrix} = \begin{pmatrix} \frac{vx}{\sqrt{vx^2+vy^2}} \\ \frac{vy}{\sqrt{vx^2+vy^2}} \end{pmatrix} \quad (2.31)$$

gives for the partial derivatives:

$$\bar{V}_u = \begin{pmatrix} \bar{v}x_u \\ \bar{v}y_u \end{pmatrix} = \begin{pmatrix} \frac{-vy \cdot (vx \cdot vy_u - vy \cdot vx_u)}{(vx^2+vy^2)^{\frac{3}{2}}} \\ \frac{vx \cdot (vx \cdot vy_u - vy \cdot vx_u)}{(vx^2+vy^2)^{\frac{3}{2}}} \end{pmatrix} \quad (2.32)$$

$$\bar{V}_v = \begin{pmatrix} \bar{v}x_v \\ \bar{v}y_v \end{pmatrix} = \begin{pmatrix} \frac{-vy \cdot (vx \cdot vy_v - vy \cdot vx_v)}{(vx^2+vy^2)^{\frac{3}{2}}} \\ \frac{vx \cdot (vx \cdot vy_v - vy \cdot vx_v)}{(vx^2+vy^2)^{\frac{3}{2}}} \end{pmatrix}. \quad (2.33)$$

(2.12), (2.18), (2.31), (2.32) and (2.33) yield

$$\kappa = \kappa^{[0]} = -\bar{v}x_v + \bar{v}y_u \quad (2.34)$$

$$\kappa^{\left[\frac{\pi}{2}\right]} = \bar{v}x_u + \bar{v}y_v. \quad (2.35)$$

(2.5), (2.22), (2.34) and (2.35) give

$$\kappa_{\max} = \sqrt{\bar{v}x_u^2 + \bar{v}y_u^2 + \bar{v}x_v^2 + \bar{v}y_v^2}. \quad (2.36)$$



Classical vector analysis provides two fundamental measures of the rate of the change of a vector field: the *divergence* and the *curl* (see [3] or any standard text book on vector analysis for details). The divergence and the curl of a vector field consider the magnitude of vectors in the vector field. Nevertheless, using the concepts of normalized and perpendicular vector fields, we are able to write the vector field curvature in terms of vector field divergence.

Given a vector field  $V = (vx, vy)^T$ , the divergence of  $V$  is defined as (see [3]):

$$\operatorname{div}(V) := vx_u + vy_v \quad (2.37)$$

and has the following geometrical meaning: considering a small area (volume for 3D vector fields) element in the flow related to the vector field, the area (volume) of this element will change during the flow. The divergence is a measure of how much the area (volume) of such an element changes during a time  $dt$ . A positive divergence means increasing of the area (volume), a negative divergence means decreasing of the area (volume).

For the perpendicular vector field  $V^{[\frac{\pi}{2}]} = (-vy, vx)^T$  we obtain

$$\operatorname{div}(V^{[\frac{\pi}{2}]}) = -vy_u + vx_v. \quad (2.38)$$

Now (2.34), (2.35), (2.37) and (2.38) give

$$\kappa = \kappa^{[0]} = -\operatorname{div}(\overline{V}^{[\frac{\pi}{2}]}) \quad (2.39)$$

$$\kappa^{[\frac{\pi}{2}]} = \operatorname{div}(\overline{V}). \quad (2.40)$$

## 2.5 Curvature Behavior Around Critical Points

In this section we want to discuss a property of the vector field curvature around critical points. This property –formulated in theorem 1– is one of the foundations of using the vector field curvature for the visualization of vector fields (treated in the next chapter).

We consider the normalized vector field  $\overline{V} = (\overline{vx}, \overline{vy})^T$  of the vector field  $V$ , given by (2.2). Then we know  $-1 \leq \overline{vx} \leq 1$ ,  $-1 \leq \overline{vy} \leq 1$  and  $\overline{vx}^2 + \overline{vy}^2 = 1$ . Furthermore, we know (2.34) and (2.35).

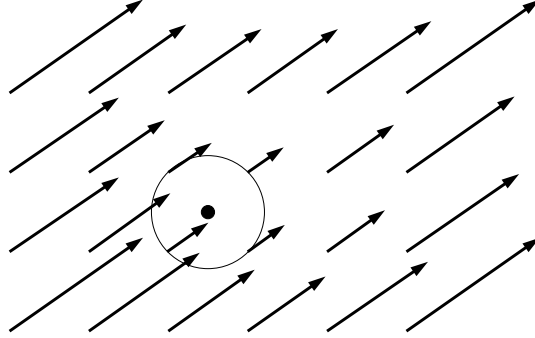


Figure 2.3: Degenerate critical point

Let  $(u_0, v_0)$  be a critical point of  $V$ . We consider a small circle with radius  $r > 0$  around  $(u_0, v_0)$ . For every point of this circle,  $\bar{V}$  defines a direction  $(\bar{v}\bar{x}, \bar{v}\bar{y})^T$ . We define the extreme values for  $\bar{v}\bar{x}$  for all points on the circle:

$$\bar{v}\bar{x}_{\min}(r) := \min \left\{ \|\bar{v}\bar{x}(u, v)\| : \sqrt{(u - u_0)^2 + (v - v_0)^2} = r \right\} \quad (2.41)$$

$$\bar{v}\bar{x}_{\max}(r) := \max \left\{ \|\bar{v}\bar{x}(u, v)\| : \sqrt{(u - u_0)^2 + (v - v_0)^2} = r \right\}. \quad (2.42)$$

Now we define:

**Definition 3** *The critical point  $(u_0, v_0)$  is degenerate iff*

$$\lim_{r \rightarrow 0+0} \bar{v}\bar{x}_{\min}(r) = \lim_{r \rightarrow 0+0} \bar{v}\bar{x}_{\max}(r) \quad (2.43)$$

for  $(u_0, v_0)$ .

That means that in a very small neighborhood of a degenerate critical point the directions of the vectors in the vector field do not change. Figure 2.3 gives an illustration of this definition.<sup>1</sup> We want to exclude degenerate critical points from further treatment.

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<sup>1</sup>From the standpoint of vector field topology (see [5]) we can say: Every critical point with an index  $\neq 0$  is non-degenerate. In fact, for every point with an index  $\neq 0$  we obtain:

$$\begin{aligned} \lim_{r \rightarrow 0+0} \bar{v}\bar{x}_{\min}(r) &= 0 \\ \lim_{r \rightarrow 0+0} \bar{v}\bar{x}_{\max}(r) &= 1. \end{aligned}$$

We consider again a small circle with radius  $r$  around the critical point  $(u_0, v_0)$ . Now we define the extreme curvature and the extreme perpendicular curvature of the vector field for all points on the circle:

$$\kappa_{extreme}^{[0]}(r) := \max \left\{ \|\kappa^{[0]}(u, v)\| : \sqrt{(u - u_0)^2 + (v - v_0)^2} = r \right\} \quad (2.44)$$

$$\kappa_{extreme}^{[\frac{\pi}{2}]}(r) := \max \left\{ \|\kappa^{[\frac{\pi}{2}]}(u, v)\| : \sqrt{(u - u_0)^2 + (v - v_0)^2} = r \right\} \quad (2.45)$$

Furthermore, we define

$$\kappa_{extreme}(r) := \max \left\{ \kappa_{extreme}^{[0]}(r), \kappa_{extreme}^{[\frac{\pi}{2}]}(r) \right\}. \quad (2.46)$$

Now we can formulate the following

**Theorem 1** *Let  $V$  be a vector field and  $(u_0, v_0)$  be a non-degenerate critical point of  $V$ . Then the following is valid around  $(u_0, v_0)$ :*

$$\lim_{r \rightarrow 0+0} \kappa_{extreme}(r) = \infty. \quad (2.47)$$

Theorem 1 tells us that in a small neighborhood of a non-degenerate critical point the vector field curvature tends to infinity – at least for one direction in either the vector field or the perpendicular vector field.

To prove theorem 1 we have to show that  $\kappa_{\max}$  tends to infinity around the non-degenerate critical point  $(u_0, v_0)$  - at least for one direction. Equation (2.36) holds that for proving this, it is sufficient to show that at least one of the values  $\overline{v x}_u, \overline{v x}_v, \overline{v y}_u, \overline{v y}_v$  tends to infinity around  $(u_0, v_0)$ .

One more time we consider a small circle  $c$  around  $(u_0, v_0)$  with radius  $r > 0$ . Furthermore, we consider two points  $P_1$  and  $P_2$  on  $c$  which give the extreme values of  $\overline{v x}$  for all points of the circle:

$$\overline{v x}(P_1) = \overline{v x}_{\min}(r) \quad (2.48)$$

$$\overline{v x}(P_2) = \overline{v x}_{\max}(r). \quad (2.49)$$

Let  $d$  be the distance between  $P_1$  and  $P_2$ . Since  $P_1$  and  $P_2$  are on  $c$ , we know

$$d \leq 2 \cdot r. \quad (2.50)$$

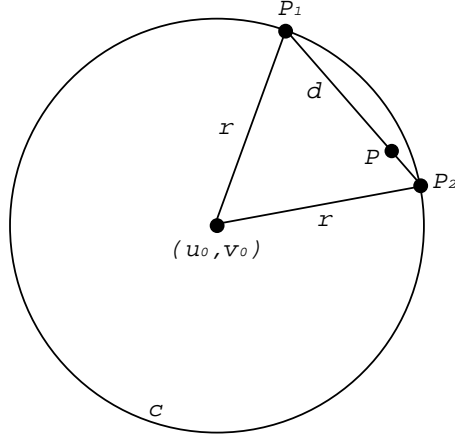


Figure 2.4: configuration for proving theorem 1

(If  $d = 2 \cdot r$ , i.e.  $P_1$  and  $P_2$  are diametral on  $c$ , we have to move either  $P_1$  or  $P_2$  a little bit on  $c$ . Doing this we have to make sure that  $\overline{v\bar{x}}(P_1) \neq \overline{v\bar{x}}(P_2)$ .) Figure 2.4 illustrates the configuration for the proof of theorem 1.

We consider the function  $\overline{v\bar{x}}$  over the line segment  $P_1P_2$ . Let  $\dot{\overline{v\bar{x}}}(P)$  be the directional derivative of  $\overline{v\bar{x}}(P)$  in the direction of  $P_1 - P_2$  for a point  $P$  on the line segment  $P_1P_2$ . Since (2.48), (2.49) and  $\overline{v\bar{x}}$  is continuous over  $P_1P_2$ , the *mean value theorem of differential calculus* (see [21]) holds: There is a point  $P$  on the line segment  $P_1P_2$  with

$$\dot{\overline{v\bar{x}}}(P) = \frac{\overline{v\bar{x}}(P_2) - \overline{v\bar{x}}(P_1)}{d}. \quad (2.51)$$

Furthermore we know that  $\dot{\overline{v\bar{x}}}(P)$  can be expressed as a linear combination of  $\overline{v\bar{x}}_u(P)$  and  $\overline{v\bar{x}}_v(P)$ :

$$\dot{\overline{v\bar{x}}}(P) = \overline{v\bar{x}}_u(P) \cdot \cos \delta + \overline{v\bar{x}}_v(P) \cdot \sin \delta \quad (2.52)$$

for some angle  $\delta$ .

Now we let the radius  $r$  of the circle  $c$  converge to 0. Then (2.50) shows that  $d$  converges to 0 as well. This, (2.51), and the fact that the critical point is non-degenerate yields that  $\dot{\overline{v\bar{x}}}(P)$  tends to infinity. This statement together with (2.52) yields that at least one of the partial derivatives of  $\overline{v\bar{x}}(P)$  converges to infinity. Thus, theorem 1 is proven  $\square$ .

Theorem 1 gives a necessary condition for a non-degenerate critical point in a vector field. Obviously, this condition is also sufficient: considering equation (2.12) and keeping in mind that the vector field is piecewise analytic, we obtain that  $\kappa$  can tend to infinity only if the denominator of (2.12) converges to 0, i.e. we have a critical point.

## 2.6 Uniqueness of Curvature Description for Vector Fields

In this section we want to show that the curvature and the perpendicular curvature of a vector field describe its normalized vector field uniquely. For doing this we prove the following

**Theorem 2** *Given are two vector fields  $V_1$  and  $V_2$  which have non-constant direction fields. If  $\kappa(V_1) = \kappa(V_2)$  and  $\kappa^{[\frac{\pi}{2}]}(V_1) = \kappa^{[\frac{\pi}{2}]}(V_2)$  then the directions of the vectors of  $V_1$  and  $V_2$  coincide in every point.*

To prove this we describe the vectors of  $V_1$  and  $V_2$  in polar coordinates, i.e. in terms of direction angle and magnitude:

$$V_1(u, v) = \begin{pmatrix} \phi(u, v) \\ m_1(u, v) \end{pmatrix} \quad (2.53)$$

$$V_2(u, v) = \begin{pmatrix} \phi(u, v) + \alpha(u, v) \\ m_2(u, v) \end{pmatrix} \quad (2.54)$$

where  $\phi$  and  $\phi + \alpha$  denote the direction angle and  $m_1$  and  $m_2$  denote the magnitudes of the vectors. Then we know from (2.28) and (2.29) about the curvatures of  $V_1$  and  $V_2$  and their partial derivatives:

$$\kappa(V_1) = \phi_u \cdot \cos \phi + \phi_v \cdot \sin \phi \quad (2.55)$$

$$\begin{aligned} \kappa_u(V_1) &= \phi_{uu} \cdot \cos \phi + \phi_u \cdot (\cos \phi)_u \\ &+ \phi_{uv} \cdot \sin \phi + \phi_v \cdot (\sin \phi)_u \\ &= \phi_{uu} \cdot \cos \phi - \phi_u^2 \cdot \sin \phi \\ &+ \phi_{uv} \cdot \sin \phi + \phi_u \cdot \phi_v \cdot \cos \phi \end{aligned} \quad (2.56)$$

$$\begin{aligned}
\kappa_v(V_1) &= \phi_{uv} \cdot \cos \phi + \phi_u \cdot (\cos \phi)_v \\
&+ \phi_{vv} \cdot \sin \phi + \phi_v \cdot (\sin \phi)_v \\
&= \phi_{uv} \cdot \cos \phi - \phi_u \cdot \phi_v \cdot \sin \phi \\
&+ \phi_{vv} \cdot \sin \phi + \phi_v^2 \cdot \cos \phi
\end{aligned} \tag{2.57}$$

$$\kappa^{[\frac{\pi}{2}]}_v(V_1) = -\phi_u \cdot \sin \phi + \phi_v \cdot \cos \phi \tag{2.58}$$

$$\begin{aligned}
\kappa^{[\frac{\pi}{2}]}_u(V_1) &= -\phi_{uu} \cdot \sin \phi - \phi_u \cdot (\sin \phi)_u \\
&+ \phi_{uv} \cdot \cos \phi + \phi_v \cdot (\cos \phi)_u \\
&= -\phi_{uu} \cdot \sin \phi - \phi_u^2 \cos \phi \\
&+ \phi_{uv} \cdot \cos \phi - \phi_u \cdot \phi_v \cdot \sin \phi
\end{aligned} \tag{2.59}$$

$$\begin{aligned}
\kappa^{[\frac{\pi}{2}]}_v(V_1) &= -\phi_{uv} \cdot \sin \phi - \phi_u \cdot (\sin \phi)_v \\
&+ \phi_{vv} \cdot \cos \phi + \phi_v \cdot (\cos \phi)_v \\
&= -\phi_{uv} \cdot \sin \phi - \phi_u \cdot \phi_v \cdot \cos \phi \\
&+ \phi_{vv} \cos \phi - \phi_v^2 \cdot \sin \phi
\end{aligned} \tag{2.60}$$

$$\kappa(V_2) = (\phi_u + \alpha_u) \cdot \cos(\phi + \alpha) + (\phi_v + \alpha_v) \cdot \sin(\phi + \alpha) \tag{2.61}$$

$$\begin{aligned}
\kappa_u(V_2) &= (\phi_{uu} + \alpha_{uu}) \cdot \cos(\phi + \alpha) + (\phi_u + \alpha_u) \cdot (\cos(\phi + \alpha))_u \\
&+ (\phi_{uv} + \alpha_{uv}) \cdot \sin(\phi + \alpha) + (\phi_v + \alpha_v) \cdot (\sin(\phi + \alpha))_u \\
&= (\phi_{uu} + \alpha_{uu}) \cdot \cos(\phi + \alpha) - (\phi_u + \alpha_u)^2 \cdot \sin(\phi + \alpha) \\
&+ (\phi_{uv} + \alpha_{uv}) \cdot \sin(\phi + \alpha) \\
&+ (\phi_u + \alpha_u) \cdot (\phi_v + \alpha_v) \cdot \cos(\phi + \alpha)
\end{aligned} \tag{2.62}$$

$$\begin{aligned}
\kappa_v(V_2) &= (\phi_{uv} + \alpha_{uv}) \cdot \cos(\phi + \alpha) + (\phi_u + \alpha_u) \cdot (\cos(\phi + \alpha))_v \\
&+ (\phi_{vv} + \alpha_{vv}) \cdot \sin(\phi + \alpha) + (\phi_v + \alpha_v) \cdot (\sin(\phi + \alpha))_v \\
&= (\phi_{uv} + \alpha_{uv}) \cdot \cos(\phi + \alpha) \\
&- (\phi_u + \alpha_u) \cdot (\phi_v + \alpha_v) \cdot \sin(\phi + \alpha) \\
&+ (\phi_{vv} + \alpha_{vv}) \cdot \sin(\phi + \alpha) + (\phi_v + \alpha_v)^2 \cdot \cos(\phi + \alpha) \quad (2.63)
\end{aligned}$$

$$\kappa_u^{[\frac{\pi}{2}]}(V_2) = - (\phi_u + \alpha_u) \cdot \sin(\phi + \alpha) + (\phi_v + \alpha_v) \cdot \cos(\phi + \alpha) \quad (2.64)$$

$$\begin{aligned}
\kappa_u^{[\frac{\pi}{2}]}(V_2) &= - (\phi_{uu} + \alpha_{uu}) \cdot \sin(\phi + \alpha) - (\phi_u + \alpha_u) \cdot (\sin(\phi + \alpha))_u \\
&+ (\phi_{uv} + \alpha_{uv}) \cdot \cos(\phi + \alpha) + (\phi_v + \alpha_v) \cdot (\cos(\phi + \alpha))_u \\
&= - (\phi_{uu} + \alpha_{uu}) \cdot \sin(\phi + \alpha) - (\phi_u + \alpha_u)^2 \cdot \cos(\phi + \alpha) \\
&+ (\phi_{uv} + \alpha_{uv}) \cdot \cos(\phi + \alpha) \\
&- (\phi_u + \alpha_u) \cdot (\phi_v + \alpha_v) \cdot \sin(\phi + \alpha) \quad (2.65)
\end{aligned}$$

$$\begin{aligned}
\kappa_v^{[\frac{\pi}{2}]}(V_2) &= - (\phi_{uv} + \alpha_{uv}) \cdot \sin(\phi + \alpha) - (\phi_u + \alpha_u) \cdot (\sin(\phi + \alpha))_v \\
&+ (\phi_{vv} + \alpha_{vv}) \cdot \cos(\phi + \alpha) + (\phi_v + \alpha_v) \cdot (\cos(\phi + \alpha))_v \\
&= - (\phi_{uv} + \alpha_{uv}) \cdot \sin(\phi + \alpha) \\
&- (\phi_u + \alpha_u) \cdot (\phi_v + \alpha_v) \cdot \cos(\phi + \alpha) \\
&+ (\phi_{vv} + \alpha_{vv}) \cdot \cos(\phi + \alpha) - (\phi_v + \alpha_v)^2 \cdot \sin(\phi + \alpha). \quad (2.66)
\end{aligned}$$

The assumption that the curvatures of  $V_1$  and  $V_2$  and the perpendicular curvatures of  $V_1$  and  $V_2$  coincide gives the system of equations

$$\left[ \begin{array}{l} \kappa(V_1) = \kappa(V_2) \\ \kappa_u(V_1) = \kappa_u(V_2) \\ \kappa_v(V_1) = \kappa_v(V_2) \\ \kappa^{[\frac{\pi}{2}]}(V_1) = \kappa^{[\frac{\pi}{2}]}(V_2) \\ \kappa_u^{[\frac{\pi}{2}]}(V_1) = \kappa_u^{[\frac{\pi}{2}]}(V_2) \\ \kappa_v^{[\frac{\pi}{2}]}(V_1) = \kappa_v^{[\frac{\pi}{2}]}(V_2) \end{array} \right] \quad (2.67)$$

with the 6 unknowns  $\alpha, \alpha_u, \alpha_v, \alpha_{uu}, \alpha_{uv}, \alpha_{vv}$ . Keeping in mind that  $\sin(\phi + \alpha) = \sin \phi \cdot \cos \alpha + \cos \phi \cdot \sin \alpha$  and  $\cos(\phi + \alpha) = \cos \phi \cdot \cos \alpha - \sin \phi \cdot \sin \alpha$ ,

we can solve this system of equations by substituting all partial derivatives of  $\alpha$ . This gives for  $\alpha_u, \alpha_v$  and  $\alpha$ :

$$\alpha_u = -\phi_v \cdot \sin \alpha + \phi_u \cdot \cos \alpha - \phi_u \quad (2.68)$$

$$\alpha_v = \phi_u \cdot \sin \alpha + \phi_v \cdot \cos \alpha - \phi_v \quad (2.69)$$

$$0 = \frac{\sin \alpha \cdot \cos \phi - \cos \alpha \cdot \sin \phi}{\cos^2 \alpha - \cos^2 \phi} \cdot ((\cos \alpha - 1) \cdot (\phi_u^2 + \phi_v^2) - \sin \alpha \cdot (\phi_{uu} + \phi_{vv})). \quad (2.70)$$

(2.70) has 3 possible solutions:

$$\alpha = 0 \quad (2.71)$$

$$\alpha = \phi \quad (2.72)$$

$$\alpha = -2 \cdot \arctan \frac{\phi_{uu} + \phi_{vv}}{\phi_u^2 + \phi_v^2}. \quad (2.73)$$

(2.71) is the trivial solution meaning that  $V_1 = V_2$ . Now we want to show that (2.72) and (2.73) are in general not solutions of our problem.

Solving the system of equations (2.67) we considered  $\alpha, \alpha_u, \alpha_v, \alpha_{uu}, \alpha_{uv}, \alpha_{vv}$  as independent variables. Using the dependencies between them we want to exclude (2.72) and (2.73) as solutions.

**a)** concerning (2.72), converging  $\alpha$  to  $\phi$  transforms (2.70) to

$$0 = \frac{(\cos \alpha - 1) \cdot (\phi_u^2 + \phi_v^2) - \sin \alpha \cdot (\phi_{uu} + \phi_{vv})}{\cos \phi \cdot \sin \phi}. \quad (2.74)$$

This gives (2.73) as solution for  $\alpha$ . Therefore, (2.72) is reduced to (2.73).

**b)** concerning (2.73), we use the abbreviations

$$q := \phi_{uu} + \phi_{vv} \quad (2.75)$$

$$r := \phi_u^2 + \phi_v^2. \quad (2.76)$$

Applying basic rules of trigonometry and differential calculus, we obtain from (2.73):



$$\sin \alpha = -2 \cdot \frac{q \cdot r}{q^2 + r^2} \quad (2.77)$$

$$\cos \alpha = \frac{q^2 - r^2}{q^2 + r^2} \quad (2.78)$$

$$\alpha_u = 2 \cdot \frac{q \cdot r_u - r \cdot q_u}{q^2 + r^2} \quad (2.79)$$

$$\alpha_v = 2 \cdot \frac{q \cdot r_v - r \cdot q_v}{q^2 + r^2}. \quad (2.80)$$

Applying this to (2.68) and (2.69), we obtain two conditions for  $\phi$ :

$$\phi_u \cdot q \cdot (q^2 + r^2) = -(q \cdot (q \cdot r_u - r \cdot q_u) + r \cdot (q \cdot r_v - r \cdot q_v)) \quad (2.81)$$

$$\phi_v \cdot q \cdot (q^2 + r^2) = r \cdot (q \cdot r_u - r \cdot q_u) - q \cdot (q \cdot r_v - r \cdot q_v). \quad (2.82)$$

This means that (2.73) can be a solution of our problem only if  $\phi$  (and therefore the vector field  $V_1$ ) satisfies (2.81) and (2.82). In general, vector fields do not have this property. (In fact, all vector fields considered in this paper do not satisfy (2.81) and (2.82).) Thus, for all those vector fields theorem 2 is proven.  $\square$

Theorem 2 has an interesting consequence: the curvature and the perpendicular curvature of a vector field  $V$  together contain all information about the directions of the vectors in  $V$ . Therefore, curvature and perpendicular curvature contain all information about the topology of a vector field.

## 2.7 Linear Vector Fields

Linear vector fields are a special case of the general vector fields considered in this paper. They are used in many of practical applications: practical CFD-data usually contains the vector information for sampled points. Between those points linear (or sometimes bilinear) interpolation of the vector components is performed. The result is a piecewise linear vector field.

Although linear vector fields can be expressed in a very simple form (so simple that even a closed formula of the tangent curves exists), they provide a variety of different topologies. See [10], [11] and [15] for a classification

of linear vector fields. A general description of vector field topology can be found in [5].

On the other hand, the number of topologies describable by linear vector fields is limited in comparison to general vector fields: the piecewise linearization of a general vector field might lead to a loss of the original topology.

In this section we want to explore some properties of the curvature of linear vector fields.

A linear vector field  $V$  can be described in the form

$$V = u \cdot \mathbf{a} + v \cdot \mathbf{b} \quad (2.83)$$

where  $\mathbf{a}$  and  $\mathbf{b}$  are vector constants. (Usually  $V$  also contains a constant term, but a simple translation transforms  $V$  into (2.83)).

### Critical points and degeneracies:

$V$  has a critical point in  $(0, 0)$ . For this critical point we have

**Lemma 2** *Given is a linear vector field described by (2.83). The critical point is degenerate (using definition 3) iff the Jacobian  $j$  of  $V$  satisfies  $j := \det[\mathbf{a}, \mathbf{b}] = 0$ .*

**Proof:** If  $j = 0$ , then  $vx$  and  $vy$  are linear dependent. That means that  $(vx, vy)^T$  has always the same (or the opposite) direction, also around critical points. Therefore, the critical point is degenerate.

If  $(0, 0)$  is degenerate then the directions of the vectors do not change in a small neighborhood of  $(0, 0)$ . Since  $V_u$  and  $V_v$  are constant for linear vector fields, this means that the directions of the vectors do not change in the entire vector field. Therefore  $vx$  and  $vy$  are linearly dependent, which causes  $j = 0$   $\square$ .

Linear vector fields described by (2.83) with  $j \neq 0$  have one and only one critical point: the point  $(0, 0)$ .

### Linearity of the radius of curvature along a ray from $(0, 0)$ :

Given is a non-degenerate vector field  $V$  described by (2.83). We want to express the domain of  $V$  in terms of polar coordinates  $r_c$  and  $\mu$ :

$$u = r_c \cdot \cos \mu \quad (2.84)$$

$$v = r_c \cdot \sin \mu. \quad (2.85)$$

This gives for  $V$ :

$$V = r_c \cdot (\cos \mu \cdot \mathbf{a} + \sin \mu \cdot \mathbf{b}) \quad , \quad V_u = \mathbf{a} \quad , \quad V_v = \mathbf{b}. \quad (2.86)$$

Inserting (2.86) into (2.12), we obtain the following

**Theorem 3** *Given is a linear vector field with a non-degenerate critical point. We consider an arbitrary ray with its origin in the critical point. Then one of the following two statements is true:*

- *the curvature of the vector field is zero for all points on the ray (except for the critical point itself),*
- *the radius of curvature of the points on the ray is proportional to their distance from the critical point (except for the critical point itself).  $\square$*

Figure 3.2 illustrates this theorem.

#### Duality:

In section 2.6 we have shown that the curvature and the perpendicular curvature of a vector field together contain all information about the directions of the vectors in the original vector field. For the special case of linear vector fields we can even make a further statement:

**Theorem 4** *Let  $V = (vx, vy)^T$  be a linear vector field with a non-degenerate critical point, and let  $\kappa$  and  $\kappa^{[\frac{\pi}{2}]}$  be the curvature and the perpendicular curvature of  $V$ . Furthermore, the vector field  $VV$  is defined by  $VV = (\kappa, \kappa^{[\frac{\pi}{2}]})^T$ .  $\kappa\kappa$  and  $\kappa\kappa^{[\frac{\pi}{2}]}$  are the curvature and the perpendicular curvature of  $VV$ . Then*

$$\det \begin{bmatrix} vx & \kappa\kappa \\ vy & \kappa\kappa^{[\frac{\pi}{2}]} \end{bmatrix} = 0. \quad (2.87)$$

The proof of theorem 4 is a simple exercise in algebra: taking (2.83) for  $V$  and computing  $\kappa$  and  $\kappa^{[\frac{\pi}{2}]}$  by (2.12) and (2.18) gives  $VV$ . Applying (2.12) and (2.18) again to  $VV$  yields  $\kappa\kappa$  and  $\kappa\kappa^{[\frac{\pi}{2}]}$ . Then it's easy to check the assumption.  $\square$

Theorem 4 has the following consequence: the two curvatures of a linear vector field, considered as another vector field, yield a new curvature and perpendicular curvature field. Those fields together considered as a vector

field always have the same direction as the original vector field  $V$ .

**Remark 2:**  $VV$  produces a critical point only where  $V$  has a critical point.  $(\kappa\kappa, \kappa\kappa^{\frac{\pi}{2}})^T$  produces a critical point only where  $VV$  has a critical point. Therefore,  $V$  and  $(\kappa\kappa, \kappa\kappa^{\frac{\pi}{2}})^T$  have the same critical points.

**Remark 3:** For a general vector field  $V = (vx, vy)^T$ , the necessary and sufficient condition for the duality property described in theorem 4 is

$$vy \cdot \left[ (hu, hv) \cdot M_x \cdot \begin{pmatrix} hu \\ hv \end{pmatrix} \right] = vx \cdot \left[ (hu, hv) \cdot M_y \cdot \begin{pmatrix} hu \\ hv \end{pmatrix} \right]$$

where

$$hu = \det[V, V_u] \quad , \quad hv = \det[V, V_v] \quad ,$$

$$M_x = \begin{bmatrix} vx_{vv} & -vx_{uv} \\ -vx_{uv} & vx_{uu} \end{bmatrix} \quad , \quad M_y = \begin{bmatrix} vy_{vv} & -vy_{uv} \\ -vy_{uv} & vy_{uu} \end{bmatrix} .$$

Since the second partial derivatives vanish for linear vector fields they always satisfy this condition.

## 2.8 3D Vector Fields

The comprehensive treatment of the curvature of 3D vector fields is not the subject of this paper. Nevertheless we want to mention some properties which are simple generalizations of the 2D case.

Let  $V = (vx(u, v, w), vy(u, v, w), vz(u, v, w))^T$  be a 3D vector field. Furthermore, let  $\dot{L}(P)$ ,  $\ddot{L}(P)$  and  $\ddot{\ddot{L}}(P)$  be the derivative vectors of the tangent curve through  $P \sim (u, v, w)$ . Then we know:

$$\dot{L} = V \tag{2.88}$$

$$\ddot{L} = vx \cdot V_u + vy \cdot V_v + vz \cdot V_w \tag{2.89}$$

$$\ddot{\ddot{L}} = vx \cdot \ddot{\ddot{L}}_u + vy \cdot \ddot{\ddot{L}}_v + vz \cdot \ddot{\ddot{L}}_w. \tag{2.90}$$

Using this we can easily compute curvature and torsion of the tangent curves (see [4]):

$$\kappa = \frac{\|\dot{\vec{L}} \times \ddot{\vec{L}}\|}{\|\dot{\vec{L}}\|^3} \quad (2.91)$$

$$\tau = \frac{\det[\dot{\vec{L}}, \ddot{\vec{L}}, \ddot{\vec{L}}]}{\|\dot{\vec{L}} \times \ddot{\vec{L}}\|^2}. \quad (2.92)$$

The concept of perpendicular vector fields can not be generalized in a simple way: a perpendicular of a 3D vector field is not uniquely defined.

A 2D vector field  $V$  defines uniquely a family of curves which are perpendicular to the vectors of  $V$  in every point. (In fact, these curves are the tangent curves of  $V^{\left[\frac{\pi}{2}\right]}$ ). In 3D the analogon of those 2D curves is a family of surfaces. These surfaces are defined by demanding that their normals have the same directions as the vectors of the 3D vector field  $V$  in every point. We want to call these surfaces *perpendicular surfaces of  $V$* .

Perpendicular surfaces of a 3D vector field  $V$  do not intersect each other (except for critical points, i.e. points with  $\|V\| = 0$ ). For every point  $P \in \mathbb{E}^3$  there is one and only one perpendicular surface through it (except for critical points). So we can define the *Gaussian curvature  $K$*  and the *mean curvature  $H$*  of the perpendicular surface for every point in the 3D space. Applying the results for the curvature of 3D tangent curves and the classical definitions of Gaussian and mean curvature, we obtain for  $K$  and  $H$ :

$$H = \frac{h}{2 \cdot \|V\|^3} \quad (2.93)$$

$$K = \frac{k}{4 \cdot \|V\|^4} \quad (2.94)$$

where

$$\begin{aligned} h &= vx \cdot (V \cdot V_u) + vy \cdot (V \cdot V_v) + vz \cdot (V \cdot V_w) \\ &- \|V\|^2 \cdot (vx_u + vy_v + vz_w) \end{aligned}$$

and

$$\begin{aligned}
k = & \left( vx \cdot (vy_w + vz_v) + vy \cdot (vx_w + vz_u) + vz \cdot (vx_v + vy_u) \right)^2 \\
& - 2 \cdot \left( vx^2 \cdot (vy_w + vz_v)^2 + vy^2 \cdot (vx_w + vz_u)^2 + vz^2 \cdot (vx_v + vy_u)^2 \right) \\
& + 4 \cdot (vx^2 \cdot vy_v \cdot vz_w + vy^2 \cdot vx_u \cdot vz_w + vz^2 \cdot vx_u \cdot vy_v) \\
& - 4 \cdot vx \cdot vy \cdot vz_w \cdot (vx_v + vy_u) \\
& - 4 \cdot vx \cdot vz \cdot vy_v \cdot (vx_w + vz_u) \\
& - 4 \cdot vy \cdot vz \cdot vx_u \cdot (vy_w + vz_v).
\end{aligned}$$

**Remark 4:** The mean curvature  $H$  can also be written in terms of vector field divergence:

$$H = -\frac{\text{div}(\bar{V})}{2} \quad (2.95)$$

where

$$\bar{V} = \frac{V}{\|V\|}.$$

A linear 3D vector field is defined by

$$V(u, v, w) = u \cdot \mathbf{a} + v \cdot \mathbf{b} + w \cdot \mathbf{c} + \mathbf{d} \quad (2.96)$$

where  $\mathbf{a}$ ,  $\mathbf{b}$ ,  $\mathbf{c}$  and  $\mathbf{d}$  are 3D vector constants.

For 3D linear vector fields theorem 3 is valid. Also for the torsion we obtain a theorem similar to 3: we only have to replace "curvature" by "torsion" and "radius of curvature" by "1 / torsion".

## Chapter 3

# Curvature and Vector Field Visualization

In this chapter we want to apply the results of chapter 2 for developing a visualization technique for vector fields. Examples of this technique are shown.

### 3.1 Previous Work and Classification

The visualization of vector fields has become one of the main topics in scientific visualization: CFD-data is usually given as vector fields, their visualization may provide new information about many processes in nature, science and technology.

Several techniques for visualizing a vector field have been developed. A survey of visualization techniques for vector fields can be found in [24]. In order to visualize vector fields, one has to solve one general problem: usually a vector field (2D and 3D) contains more information than is visualizable on a screen. So we have to pick out the parts of information which are most important for our application. Then we have to choose (or create) a suitable visualization technique which emphasizes the selected kind of information.

Since every technique for visualizing a vector field can only handle a part of the information in the vector field, we can make the following statement:

There is no general visualization technique for vector fields which is suitable for all applications. Every technique is appropriate only for a particular class of applications.

We want to consider only static visualizations of a steady flow, i.e. we have one fixed picture for the time independent vector field. Based on the question of how the information of the vector field is reduced we can classify the vector field visualization techniques in the following classes (where combinations of different classes are possible):

**1)** Pick out single points of the vector field and visualize local properties in these points as icons. These icons are usually arrow-like objects and contain direction and magnitude of the vectors in the particular points. In [18] a more complex icon for visualizing more local properties (divergence and shear of the vector field, curvature and torsion of the tangent curves) is used. [18] seems to be the first paper to use the curvature of tangent curves for vector field visualization. The first order approximation of a 3D vector field is taken to compute curvature and torsion. The obtained curvature coincides with the curvature of the actual (unapproximated) vector field. The torsion differs because the first order approximation of a vector field considers only its first order partial derivatives.

**2)** Pick out a single property of the vector field and visualize it over the entire domain of the vector field. These single properties can be scalars like magnitude or direction angle of the vector field. The visualization can be done by color coding or contouring (see [9]).

**3)** Visualize the tangent curves of the vector field. Tangent curves are a powerful tool for visualizing vector fields. The knowledge of tangent curves implies the knowledge of the directions of the vectors in the vector field. The visualization of tangent curves creates two major problems:

-a) It must be decided how many tangent curves are visualized. Too many curves lead to a confusing display. On the other hand, there must be as many visualized tangent curves as the user needs to infer the behavior of the remaining curves.

-b) As mentioned in chapter 2, tangent curves can in general not be described as parametric curves but only as the solution of a system of differential equations. Their visualization requires a numerical solution of those equations. Several approaches for a numerical integration of tangent curves are presented



in [2], [6], [7], [10], [11], [15], [24] and [26].

The use of topological concepts is an approach for solving problem a). In [10], [11] and [15] the critical points of the vector field are detected and classified. These points are connected by particular tangent curves, called separation curves. Unfortunately, the classification of the critical points works only for the first order approximation of vector fields. If the first order approximation changes the topology the method might give us a wrong image of the vector field.

Another approach for solving problem a) can be found in [26]. Here, a line integral convolution technique is used for visualizing the vector field. Since this technique is also based on the numerical integration of tangent curves, the risk of destroying the topology of the vector field remains.

## 3.2 The Technique and Examples

The visualization technique for vector fields presented in this section is a combination of the classes 2) and 3) from section 3.1. We want to use the power of tangent curves but avoid the problems a) and b) of class 3). We achieve this by visualizing not the tangent curves directly but one of their most characteristic properties: their curvature. The curvature of the tangent curves reflects important properties of the curves, and therefore yields information about the behavior of the entire vector field. Generally, turbulences in the flow of fluids and gases lead to extremely high and frequently changing curvatures of the tangent curves.

Using the results from chapter 2, we can compute the curvature and the perpendicular curvature of a vector field and visualize these two scalar fields by color coding. The color coding map is shown in figure 3.1.

The curvature  $\kappa$  (which can lie anywhere between  $-\infty$  and  $\infty$ ) is "normalized" to  $\kappa_{norm}$  in the interval  $\langle -1, 1 \rangle$  using the equation

$$\kappa_{norm} = \text{sgn}(\kappa) \cdot (1 - e^{-\|\kappa\| \cdot con}).$$

The same is done for  $\kappa^{[\frac{\pi}{2}]}$ . The positive value  $con$  can be considered as the contrast of the visualization. Decreasing  $con$  leads to a darker picture but emphasizes the critical points.  $con$  should be chosen interactively.

We consider some **examples**.

**Figure 3.2** shows the visualization of a linear vector field. The critical point is a saddle point for both the vector field and the perpendicular vector field. The upper left picture shows the direction of the vectors for some sampled points. The upper right picture is the curvature plot. The lower two pictures show the same for the perpendicular vector field. The critical point of the vector field can be detected as a highlight in the curvature plots. As shown in theorem 3, the curvature is inversely proportional to the distance to the critical point along a ray from the critical point.

**Figure 3.3** shows the visualization of the vector field

$$V(u, v) = \begin{pmatrix} 3 \cdot u \cdot v^2 - u \\ 3 \cdot v \cdot u^2 - v \end{pmatrix}$$

in the range  $(u = \langle -1.5, 1.5 \rangle, v = \langle -1.5, 1.5 \rangle)$ . Again, the upper left picture shows the directions of the vectors for some sampled points, and the upper right picture is the curvature plot of the vector field. The lower two pictures show the same for the perpendicular vector field. This vector field has 5 critical points  $(0, 0), (-\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}), (-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}), (\frac{\sqrt{3}}{3}, -\frac{\sqrt{3}}{3}), (\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3})$ . They can be easily detected as highlights in the curvature plots. The critical point  $(0, 0)$  is an example of a non-degenerate critical point which does not produce a highlight in the curvature plot. But then – as shown in theorem 1 – it must produce a highlight in the perpendicular curvature plot.

**Figure 3.4** shows the vector field

$$\begin{aligned} vx &= - 0.232875 \cdot u^2 + 0.037546 \cdot u \cdot v + 0.037546 \cdot v^2 \\ &+ 0.051511 \cdot u - 0.302699 \cdot v - 0.103209 \\ vy &= - 1.029676 \cdot u^2 - 0.213010 \cdot u \cdot v + 0.246278 \cdot v^2 \\ &+ 0.687847 \cdot u - 0.144779 \cdot v + 0.143656 \end{aligned}$$

in the range  $(u = \langle -1.5, 1.5 \rangle, v = \langle -1.5, 1.5 \rangle)$ . This vector field was taken from [15]. The critical points are well detectable in the curvature plots.

**Figure 3.5** shows the vector field

$$V(u, v) = \begin{cases} \begin{pmatrix} \operatorname{sgn}(u) \cdot (-2 \cdot v^2 + u^2) \\ \operatorname{sgn}(v) \cdot (-v^2) \end{pmatrix} & \text{if } \|v\| < \|u\| \\ \begin{pmatrix} \operatorname{sgn}(u) \cdot (-u^2) \\ \operatorname{sgn}(v) \cdot (-2 \cdot u^2 + v^2) \end{pmatrix} & \text{otherwise} \end{cases}$$

in the range ( $u = \langle -1.5, 1.5 \rangle, v = \langle -1.5, 1.5 \rangle$ ). This vector field has a higher order critical point in  $(0, 0)$  - a saddle point with 4 pairs of tangent curves through it. It should therefore be clear that applying visualization methods based on first order approximation of the vector field would not lead to satisfactory results for this example.

The vector field visualized in **figure 3.6** is obtained by bilinear interpolation of the vectors on a regular  $4 \times 4$  grid. We obtain gaps in the curvature plots between the grid cells. (In this figure the arrow plots also consider the magnitudes of the vectors in the vector field.) In general, piecewise (bi-)linear interpolation produces tangent curves which are not curvature continuous at the borders of the grid cells.

**Figure 3.7** shows the flow of water in the bay area of the Baltic Sea near Greifswald, Germany (Greifswalder Bodden). This data is taken from [27]. The bay covers an area of  $23 \times 26$ km. The maximal depth of the water is 12m. The flow in this shallow water can be considered as a steady 2D flow. The vectors of the sample points on a regular  $115 \times 103$  grid are obtained by a numerical simulation. Between the grid points a bilinear interpolation is applied.

The curvature visualization of this vector field shows many critical points, i.e. turbulences in the flow. There is no risk of missing certain critical points. The border lines of the grid cells appear as discontinuities in the curvature visualization.

### 3.3 Assessment of the Technique

The vector field visualization technique for 2D vector fields introduced in section 3.2 has the following properties:

- The visualization is "exact", we don't have to apply any numerical solution method.
- non-degenerate critical points can always be recognized as highlights - at least in the  $\kappa$ - or the  $\kappa^{[\frac{\pi}{2}]}$ - visualization (as shown in theorem 1). Conversely, highlights in the visualization always indicate critical points in the vector field.

– From the  $\kappa$ - and the  $\kappa^{[\frac{\pi}{2}]}$ - visualization we can uniquely infer the normalized original vector field (as shown in theorem 2). Therefore, the  $\kappa$ - and the  $\kappa^{[\frac{\pi}{2}]}$ - visualizations contain all information about the topology of the original vector field. Since we don't have to first order approximate the vector field, there is no risk of destroying its topology.

– We obtain a non-confusing visualization without overloading and ambiguities.

There are disadvantages of the technique as well:

– The curvature plot does not directly provide any information about the direction of the vector in a point – which might be useful for some applications. But if we look for areas of turbulences in CFD data sets, this disadvantage should not play an important role.

– The user is not accustomed to dealing with curvature information of tangent curves, even if this provides a great deal of information about the behavior of the vector field.

In principle, the visualization technique can also be used for 3D vector fields. To visualize the curvature of a 3D vector field we have to use methods of volume visualization.

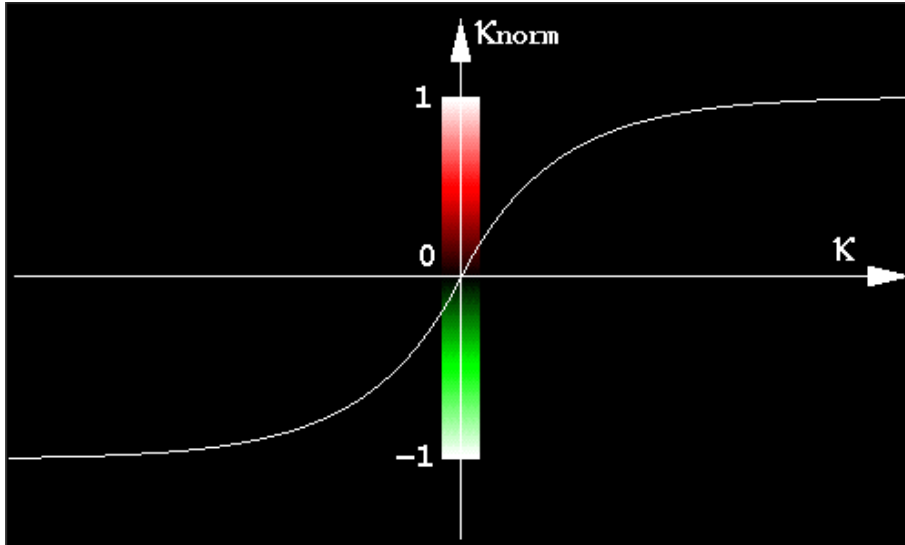


Figure 3.1: Color coding the curvature of vector fields

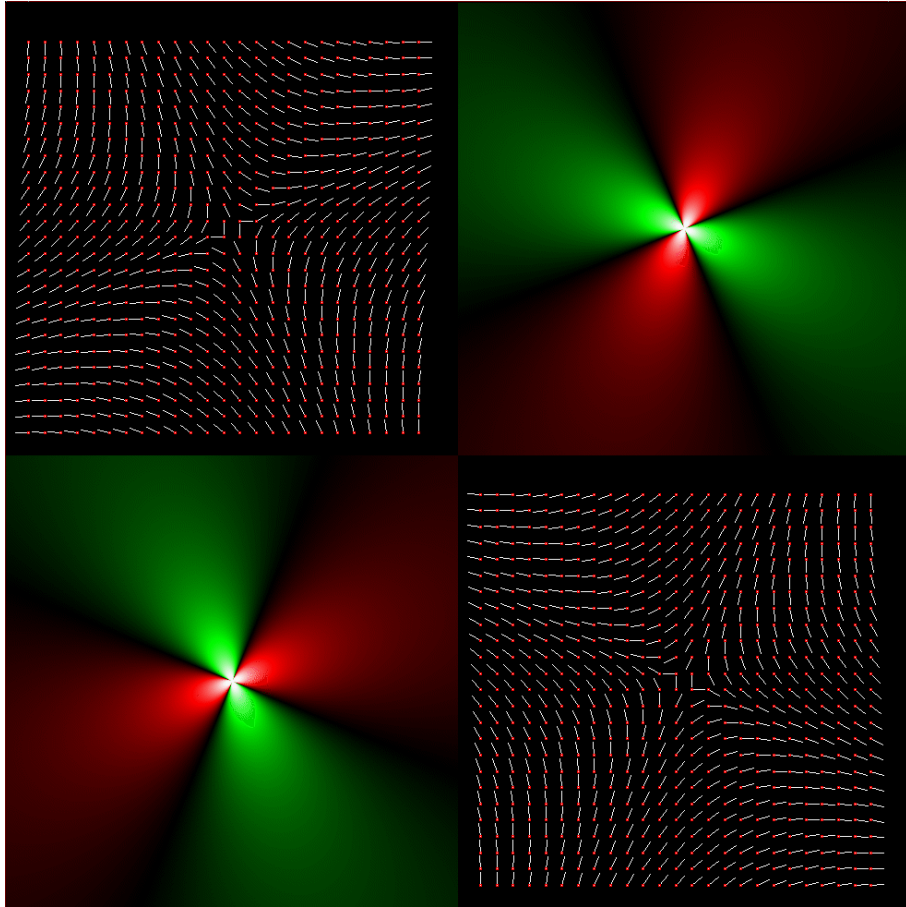


Figure 3.2: Linear vector field with saddle point

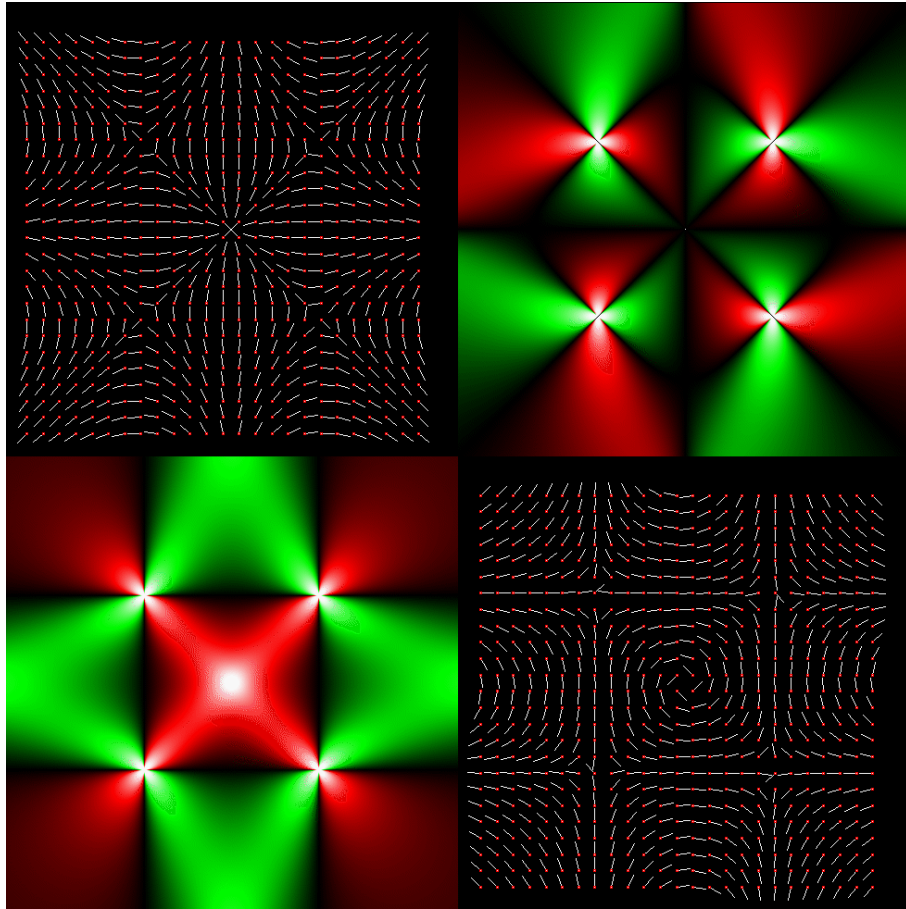


Figure 3.3: Vector field with 5 critical points

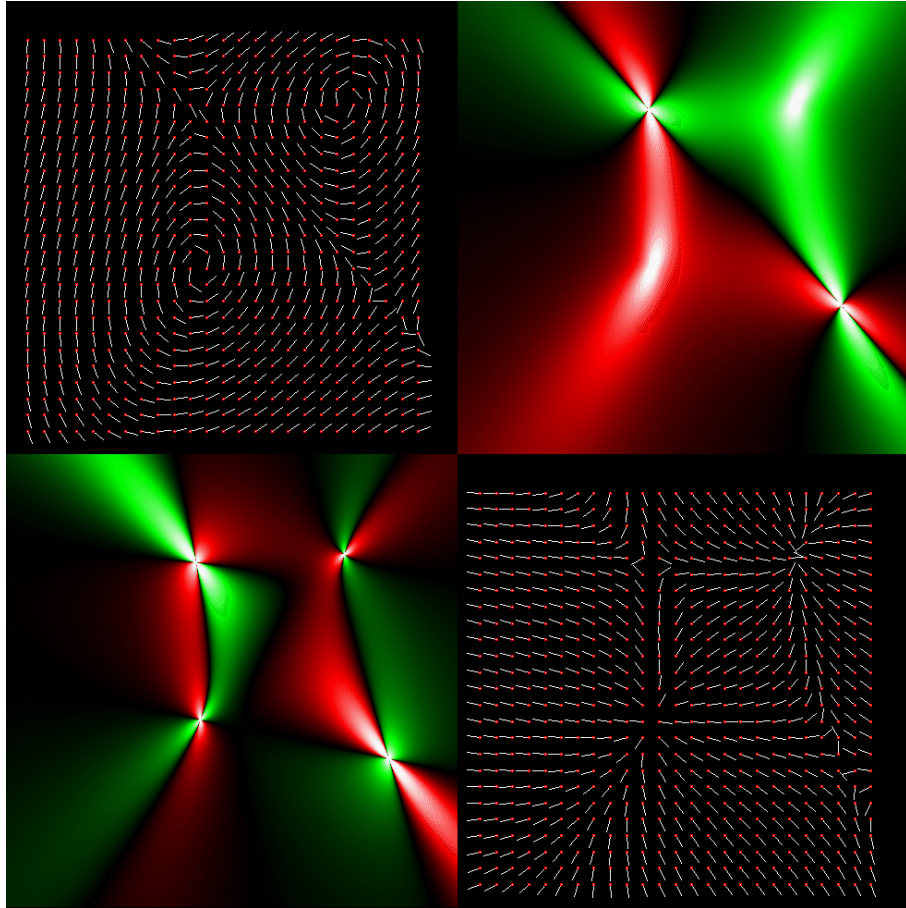


Figure 3.4: Vector field with 4 critical points



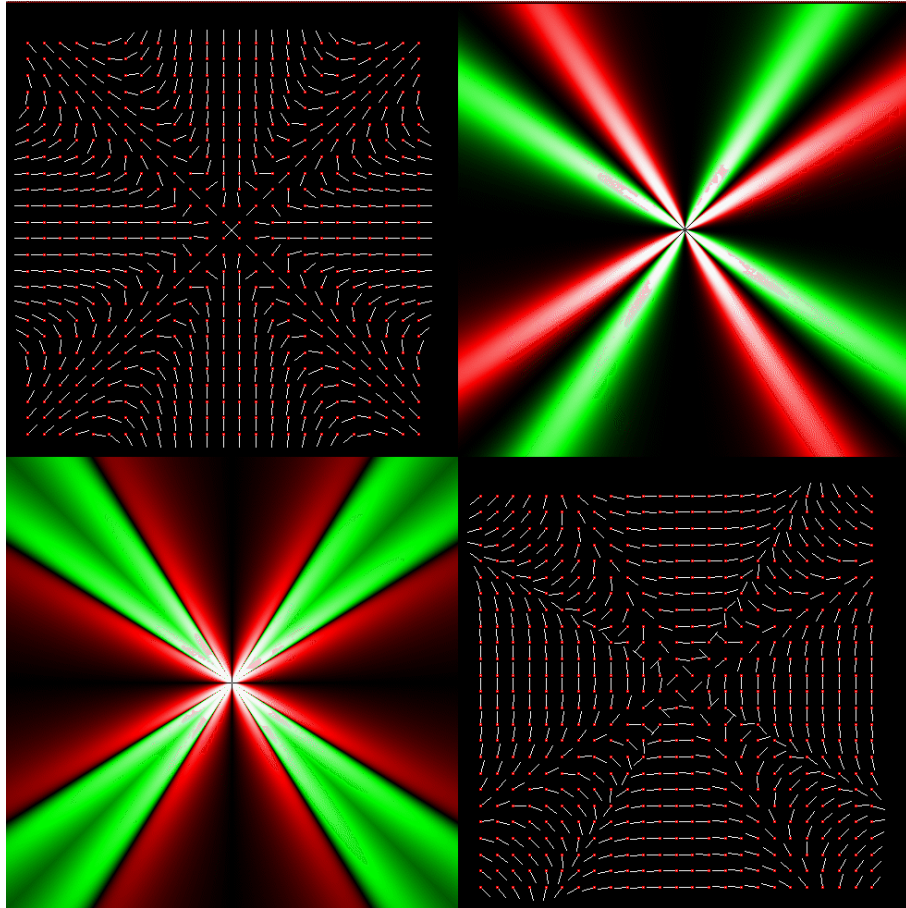


Figure 3.5: Vector field with a higher order critical point

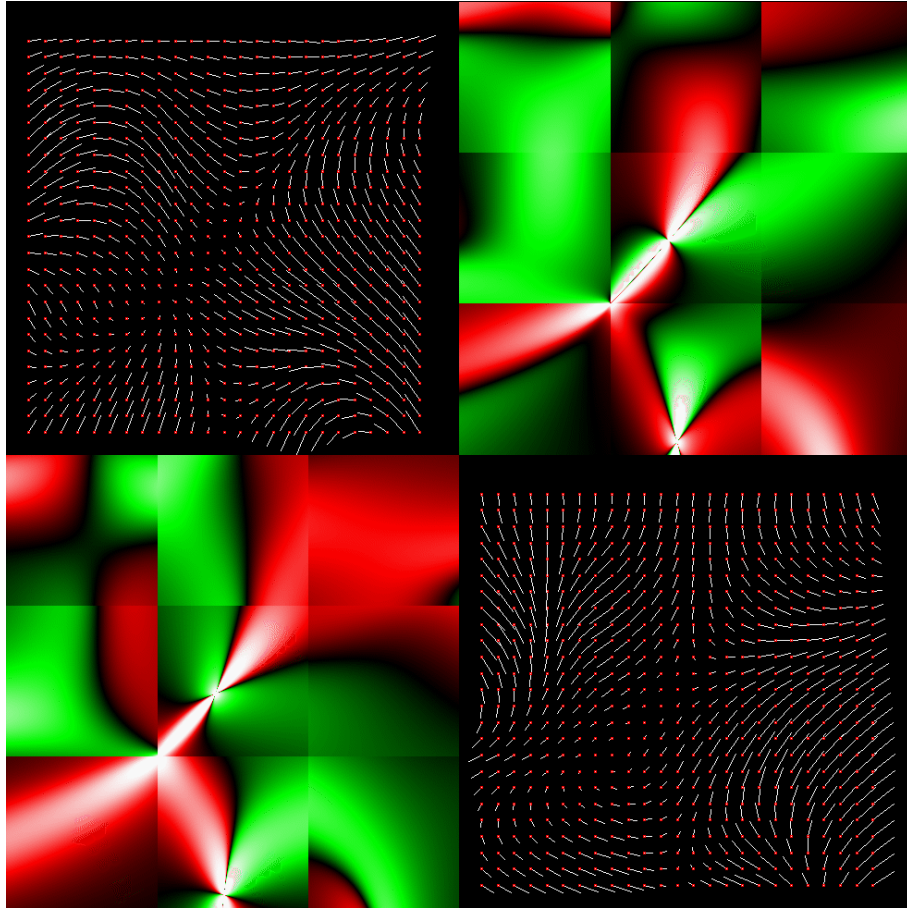


Figure 3.6: Piecewise bilinear vector field

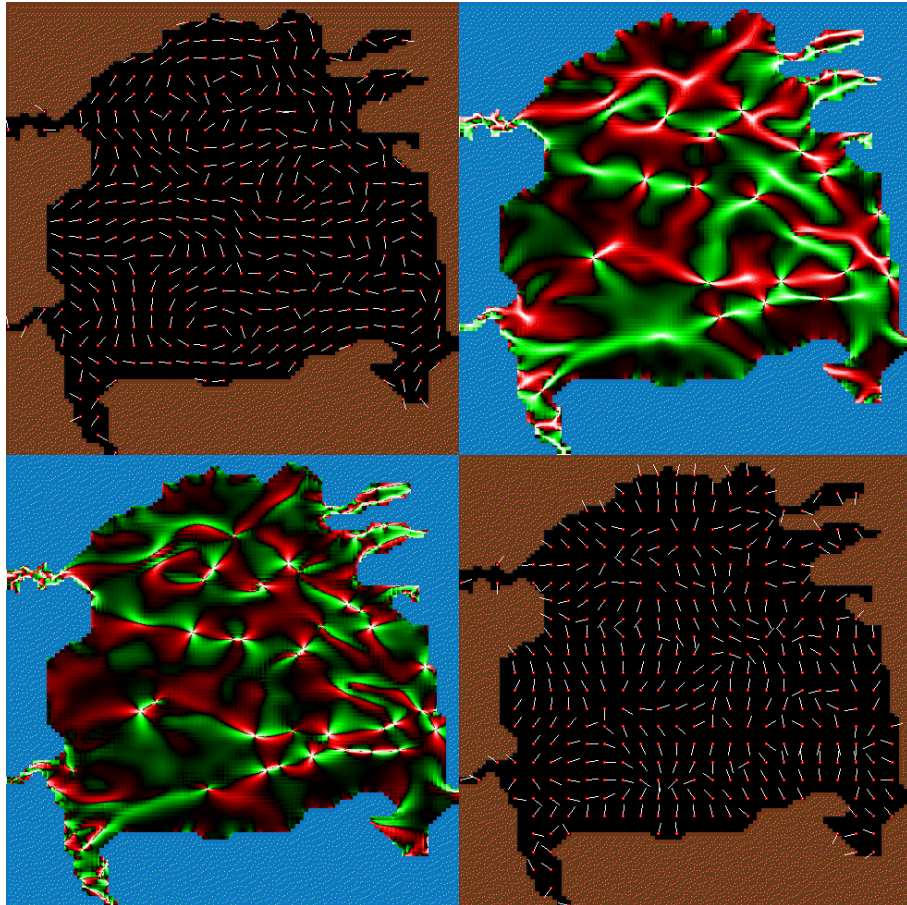


Figure 3.7: Flow in a bay area of the Baltic Sea

# Chapter 4

## Tangent Curves on Surfaces

The 2D tangent curves considered till now were located in a plane, i.e., on a special surface. In the next two chapters we want to consider tangent curves on general parametrized surfaces and we want to explore some of their properties.

### 4.1 Definitions, Notations, Abbreviations

*Tangent curves on surfaces* can be defined in two ways.

**Definition 4** Given is a surface  $\mathbf{x}(u, v) = \begin{pmatrix} x(u, v) \\ y(u, v) \\ z(u, v) \end{pmatrix}$  and a map

$W : \mathbb{E}^2 \rightarrow \mathbb{R}^3$ .  $W$  relates any point of the domain to a vector in 3D. Together with  $\mathbf{x}$ ,  $W$  can be considered as relating any point  $\mathbf{x}(u, v)$  on the surface to a 3D-vector  $W(u, v)$ .  $W$  is called vector field over the surface  $\mathbf{x}$ .

A tangent curve defined by  $W$  is a curve on the surface where the tangent vector and the projection of  $W(u, v)$  into the tangent plane of  $\mathbf{x}(u, v)$  have the same direction for any point of the curve.

See figure 4.1 for an illustration of this definition.

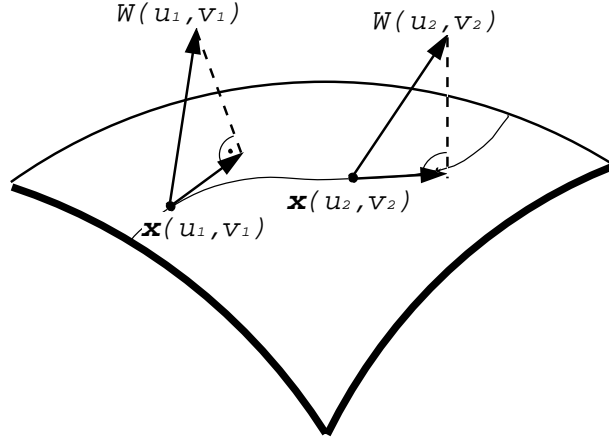


Figure 4.1: A vector field  $W$  over a surface  $\mathbf{x}$ . Shown are  $\mathbf{x}$ ,  $W$  and the projection of  $W$  in the tangent plane of  $\mathbf{x}$  for two points  $(u_1, v_1)$  and  $(u_2, v_2)$ .

**Definition 5** *Given is a surface  $\mathbf{x}$  and a 2D vector field  $V$  in the domain.  $V$  produces a family of tangent curves in the domain. The maps of these domain curves onto the surface  $\mathbf{x}$  are called tangent curves on the surface  $\mathbf{x}$ .*

Figure 4.2 illustrates this definition. The correlations between the definitions 4 and 5 are discussed in the next section.

We want to use the following notations and abbreviations:

$\dot{\mathbf{x}}(u, v)$  and  $\ddot{\mathbf{x}}(u, v)$  denote the *first and second derivative vector of the tangent curve* on  $\mathbf{x}$  through  $\mathbf{x}(u, v)$  in the point  $\mathbf{x}(u, v)$ . (Since we deal with corresponding vector fields  $W$  and  $V$  in the following, we don't distinguish in the notation of  $\dot{\mathbf{x}}$  and  $\ddot{\mathbf{x}}$  whether they are obtained from  $W$  or  $V$ .)

$\mathbf{n} = \mathbf{n}(u, v) = \frac{\mathbf{x}_u \times \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}$  is the *normalized normal vector* of the surface  $\mathbf{x}(u, v)$ .

The *normal line* of  $\mathbf{x}(u, v)$  is defined by  $\mathbf{x}(u, v) + \lambda \cdot \mathbf{n}(u, v)$ , ( $\lambda \in \mathbb{R}$ ).

Furthermore, we use the classical abbreviations

$$E = \mathbf{x}_u \cdot \mathbf{x}_u \quad (4.1)$$

$$F = \mathbf{x}_u \cdot \mathbf{x}_v \quad (4.2)$$

$$G = \mathbf{x}_v \cdot \mathbf{x}_v \quad (4.3)$$

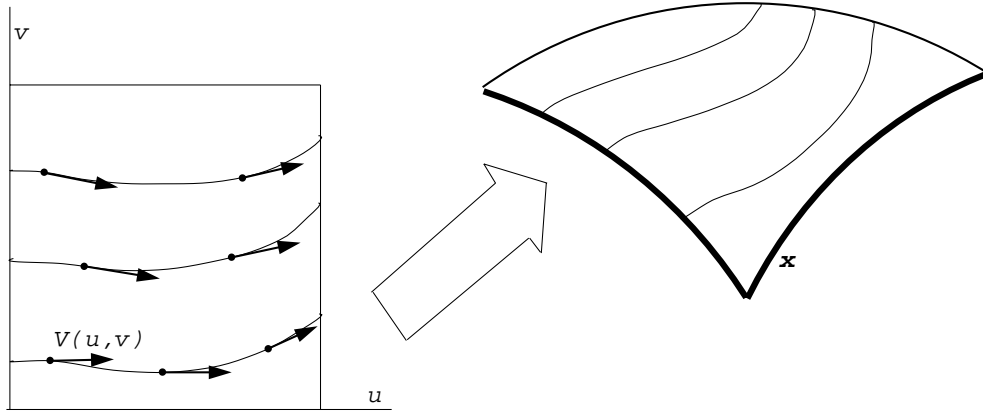


Figure 4.2: A vector field  $V$  in the domain and the mapping of its tangent curves onto the surface  $\mathbf{x}$ .

$$L = \mathbf{n} \cdot \mathbf{x}_{uu} \quad (4.4)$$

$$M = \mathbf{n} \cdot \mathbf{x}_{uv} \quad (4.5)$$

$$N = \mathbf{n} \cdot \mathbf{x}_{vv} \quad (4.6)$$

and their partial derivatives

$$E_u = 2 \cdot \mathbf{x}_u \cdot \mathbf{x}_{uu} \quad (4.7)$$

$$E_v = 2 \cdot \mathbf{x}_v \cdot \mathbf{x}_{vv} \quad (4.8)$$

$$F_u = \mathbf{x}_u \cdot \mathbf{x}_{uv} + \mathbf{x}_{uu} \cdot \mathbf{x}_v \quad (4.9)$$

$$F_v = \mathbf{x}_v \cdot \mathbf{x}_{uv} + \mathbf{x}_{vv} \cdot \mathbf{x}_u \quad (4.10)$$

$$G_u = 2 \cdot \mathbf{x}_{uv} \cdot \mathbf{x}_v \quad (4.11)$$

$$G_v = 2 \cdot \mathbf{x}_{uv} \cdot \mathbf{x}_u \quad (4.12)$$

$$L_u = \mathbf{n}_u \cdot \mathbf{x}_{uu} + \mathbf{n} \cdot \mathbf{x}_{uuu} \quad (4.13)$$

$$L_v = \mathbf{n}_v \cdot \mathbf{x}_{uu} + \mathbf{n} \cdot \mathbf{x}_{uuv} \quad (4.14)$$

$$M_u = \mathbf{n}_u \cdot \mathbf{x}_{uv} + \mathbf{n} \cdot \mathbf{x}_{uuv} \quad (4.15)$$

$$M_v = \mathbf{n}_v \cdot \mathbf{x}_{uv} + \mathbf{n} \cdot \mathbf{x}_{uvv} \quad (4.16)$$

$$N_u = \mathbf{n}_u \cdot \mathbf{x}_{vv} + \mathbf{n} \cdot \mathbf{x}_{uvv} \quad (4.17)$$

$$N_v = \mathbf{n}_v \cdot \mathbf{x}_{vv} + \mathbf{n} \cdot \mathbf{x}_{vvv} \quad (4.18)$$

## 4.2 Corresponding Vector Fields on Surfaces

**Definition 6** *Given is a surface  $\mathbf{x}$ , a 2D vector field  $V$  in the domain of  $\mathbf{x}$  and a 3D vector field  $W$  over  $\mathbf{x}$ .  $W$  and  $V$  are called corresponding referring to the surface  $\mathbf{x}$ , if the two families of tangent curves obtained from  $W$  (using definition 4) and  $V$  (using definition 5) are identical.*

Definition 6 gives reason to solve the following two problems:

- 1) Given is a surface  $\mathbf{x}$  and a domain vector field  $V$ . We look for a corresponding vector field  $W$  over  $\mathbf{x}$  as well as for  $\dot{\mathbf{x}}$  and  $\ddot{\mathbf{x}}$ .
- 2) Given is a surface  $\mathbf{x}$  and a vector field  $W$  over  $\mathbf{x}$ . We look for a corresponding domain vector field  $V$ , for  $\dot{\mathbf{x}}$  and  $\ddot{\mathbf{x}}$ .

We start with **problem 1**).

Given a curve  $[u = u(t), v = v(t)]$  in the domain, we can easily compute the map of this curve on the surface and its first and second derivative vector (see [4]):

$$\begin{aligned}\mathbf{x}(t) &= \mathbf{x}(u(t), v(t)) \\ \dot{\mathbf{x}}(t) &= (\dot{u} \cdot \mathbf{x}_u + \dot{v} \cdot \mathbf{x}_v)(t) \\ \ddot{\mathbf{x}}(t) &= (\ddot{u} \cdot \mathbf{x}_u + \ddot{v} \cdot \mathbf{x}_v + \dot{u}^2 \cdot \mathbf{x}_{uu} + 2 \cdot \dot{u} \cdot \dot{v} \cdot \mathbf{x}_{uv} + \dot{v}^2 \cdot \mathbf{x}_{vv})(t).\end{aligned}\tag{4.19}$$

Considering the domain curve as a tangent curve in the domain defined by  $V$ , we know the first and the second derivative vector in any point of the domain from section 2.2:

$$\begin{aligned}\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} &= V \\ \begin{pmatrix} \ddot{u} \\ \ddot{v} \end{pmatrix} &= vx \cdot V_u + vy \cdot V_v.\end{aligned}\tag{4.20}$$

(4.19) and (4.20) yield for the tangent curves on the surface  $\mathbf{x}$ :

$$\dot{\mathbf{x}} = vx \cdot \mathbf{x}_u + vy \cdot \mathbf{x}_v\tag{4.21}$$

$$\ddot{\mathbf{x}} = vx \cdot \dot{\mathbf{x}}_u + vy \cdot \dot{\mathbf{x}}_v\tag{4.22}$$

$$\begin{aligned}&= (vx \cdot vx_u + vy \cdot vx_v) \cdot \mathbf{x}_u \\ &+ (vx \cdot vy_u + vy \cdot vy_v) \cdot \mathbf{x}_v \\ &+ vx^2 \cdot \mathbf{x}_{uu} + 2 \cdot vx \cdot vy \cdot \mathbf{x}_{uv} + vy^2 \cdot \mathbf{x}_{vv}.\end{aligned}\tag{4.23}$$

Furthermore, a corresponding vector field to  $V$  is

$$W = \dot{\mathbf{x}}. \quad (4.24)$$

**To problem 2):**

Given the vector field  $W$  over  $\mathbf{x}$ , the first derivative vector of the tangent curve is the projection of  $W$  onto the tangent plane for every point of the surface:

$$\dot{\mathbf{x}} = W - (W \cdot \mathbf{n}) \cdot \mathbf{n}. \quad (4.25)$$

To obtain a corresponding vector field  $V$ , we have to solve the linear system of equations denoted by (4.21). This system consists of 3 equations and the two unknowns  $vx$  and  $vy$ . Since  $\dot{\mathbf{x}}$ ,  $\mathbf{x}_u$  and  $\mathbf{x}_v$  are all in one plane (namely in the tangent plane of  $\mathbf{x}$ ), we can find a solution for any regularly parametrized surface. This solution is:

$$V = \begin{pmatrix} vx \\ vy \end{pmatrix} = \frac{1}{\|\mathbf{x}_u \times \mathbf{x}_v\|} \cdot \begin{pmatrix} \det[\mathbf{n}, W, \mathbf{x}_v] \\ -\det[\mathbf{n}, W, \mathbf{x}_u] \end{pmatrix}. \quad (4.26)$$

(4.26) has reduced problem 2) to problem 1). Now we can compute  $\dot{\mathbf{x}}$  in a similar way as in problem 1). We obtain (4.22) where we know from (4.25) that

$$\begin{aligned} \dot{\mathbf{x}}_u &= W_u - (W_u \cdot \mathbf{n}) \cdot \mathbf{n} - (W \cdot \mathbf{n}_u) \cdot \mathbf{n} - (W \cdot \mathbf{n}) \cdot \mathbf{n}_u \\ \dot{\mathbf{x}}_v &= W_v - (W_v \cdot \mathbf{n}) \cdot \mathbf{n} - (W \cdot \mathbf{n}_v) \cdot \mathbf{n} - (W \cdot \mathbf{n}) \cdot \mathbf{n}_v. \end{aligned}$$

**Remark 1:** The solutions of the problems 1) and 2) described above are dual in the following sense: The corresponding vector field of the corresponding vector field of  $V$  is  $V$  as well. The corresponding vector field of the corresponding vector field of  $W$  is the projection of  $W$  onto the tangent planes of  $\mathbf{x}$ .

**Remark 2:** Another (and sometimes easier) solution for problem 2) is

$$V = \begin{pmatrix} \det[\mathbf{n}, W, \mathbf{x}_v] \\ -\det[\mathbf{n}, W, \mathbf{x}_u] \end{pmatrix}. \quad (4.27)$$

This choice of  $V$  leads to

$$\dot{\mathbf{x}} = \det[\mathbf{n}, W, \mathbf{x}_v] \cdot \mathbf{x}_u - \det[\mathbf{n}, W, \mathbf{x}_u] \cdot \mathbf{x}_v \quad (4.28)$$



which is parallel to the  $\dot{\mathbf{x}}$  from (4.25). Furthermore, we have

$$\begin{aligned} V_u &= \begin{pmatrix} \det[\mathbf{n}_u, W, \mathbf{x}_v] + \det[\mathbf{n}, W_u, \mathbf{x}_v] + \det[\mathbf{n}, W, \mathbf{x}_{uv}] \\ -\det[\mathbf{n}_u, W, \mathbf{x}_u] - \det[\mathbf{n}, W_u, \mathbf{x}_u] - \det[\mathbf{n}, W, \mathbf{x}_{uu}] \end{pmatrix} \\ V_v &= \begin{pmatrix} \det[\mathbf{n}_v, W, \mathbf{x}_v] + \det[\mathbf{n}, W_v, \mathbf{x}_v] + \det[\mathbf{n}, W, \mathbf{x}_{vv}] \\ -\det[\mathbf{n}_v, W, \mathbf{x}_u] - \det[\mathbf{n}, W_v, \mathbf{x}_u] - \det[\mathbf{n}, W, \mathbf{x}_{uv}] \end{pmatrix} \end{aligned}$$

Using (4.23) we can compute  $\ddot{\mathbf{x}}$ .

Unfortunately, this choice of  $V$  destroys the duality described in remark 1.

### 4.3 Curvature and Geodesic Curvature

Given  $\dot{\mathbf{x}}(u, v)$  and  $\ddot{\mathbf{x}}(u, v)$ , we can easily compute the curvature  $\kappa(u, v)$  of the tangent curve through  $\mathbf{x}(u, v)$  in the surface point  $\mathbf{x}(u, v)$ :

$$\kappa = \frac{\|\dot{\mathbf{x}} \times \ddot{\mathbf{x}}\|}{\|\dot{\mathbf{x}}\|^3}. \quad (4.29)$$

(4.29) denotes the curvature of a 3D space curve, therefore  $\kappa$  is always non-negative. In order to get a signed curvature, we want to compute the geodesic curvature of the tangent curves.

The *geodesic curvature*  $\kappa_g$  of a curve in the point  $\mathbf{x}(u, v)$  can be considered as the curvature of the projection of the curve into the tangent plane of  $\mathbf{x}(u, v)$ , (see figure 4.3).

Since  $\kappa_g$  is the curvature of a 2D curve we can equip it with a sign. We can compute  $\kappa_g$  by projecting  $\dot{\mathbf{x}}$  and  $\ddot{\mathbf{x}}$  into the tangent plane and taking a sign into consideration:

$$\begin{aligned} \dot{\mathbf{x}}_g &= \dot{\mathbf{x}} \\ \ddot{\mathbf{x}}_g &= \ddot{\mathbf{x}} - (\ddot{\mathbf{x}} \cdot \mathbf{n}) \cdot \mathbf{n} \end{aligned} \quad (4.30)$$

$$\kappa_g = \text{sgn}(\det[\dot{\mathbf{x}}, \ddot{\mathbf{x}}, \mathbf{n}]) \cdot \frac{\|\dot{\mathbf{x}}_g \times \ddot{\mathbf{x}}_g\|}{\|\dot{\mathbf{x}}_g\|^3} \quad (4.31)$$

Between the curvature  $\kappa$  and the geodesic curvature  $\kappa_g$  there is the following correlation (see [20]):

$$\kappa = \sqrt{\kappa_n^2 + \kappa_g^2} \quad (4.32)$$

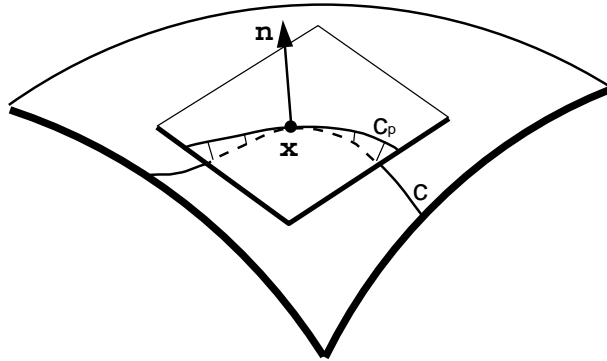


Figure 4.3: Curvature and geodesic curvature.  $c_p$  is the projection of the surface curve  $c$  into the tangent plane of  $\mathbf{x} = \mathbf{x}(u, v)$ . The geodesic curvature of  $c$  in  $\mathbf{x}(u, v)$  is the curvature of  $c_p$  in  $\mathbf{x}(u, v)$

where  $\kappa_n$  denotes the normal curvature of the surface in the direction  $\dot{\mathbf{x}}$ .

Altogether, we can make the following statement: In order to compute the curvature (the geodesic curvature respectively) of tangent curves on surfaces, we only need to know the surface, the domain vector field  $V$  and its partial derivatives  $V_u$  and  $V_v$ .

**Remark 3:** The values obtained for  $\kappa$  do not depend on the magnitudes of the vectors in  $W$  or  $V$ .

**Remark 4:** The domain vector field  $V$  has a critical point in  $(u, v)$  iff the magnitude of  $V(u, v)$  vanishes. The vector field  $W$  over  $\mathbf{x}$  has a critical point in  $(u, v)$  iff  $\mathbf{n}(u, v) \times W(u, v) = \mathbf{0}$ .

## 4.4 Rotated Vector Fields over Surfaces

In section 2.3 we have investigated rotated vector fields in the plane and their curvatures. In this section we want to expand these concepts to rotated vector fields over surfaces.

Let  $W$  and  $Y$  be two vector fields over a surface  $\mathbf{x}$ .  $W$  produces a family of tangent curves,  $Y$  produces another family of tangent curves on  $\mathbf{x}$ .  $W$  and  $Y$  are called *perpendicular relative to the surface*  $\mathbf{x}$  if the tangent curves of  $W$  and  $Y$  on  $\mathbf{x}$  are perpendicular to each other in every point of  $\mathbf{x}$ . Therefore,  $W$

and  $Y$  are perpendicular relative to  $\mathbf{x}$  if their projections  $W_p$  and  $Y_p$  into the tangent plane are perpendicular to each other for every point on the surface. So we have the following condition for perpendicularity of  $W$  and  $Y$  relative to  $\mathbf{x}$ :

$$\begin{aligned} W_p \cdot Y_p &= 0 \\ \iff (W - (W \cdot \mathbf{n}) \cdot \mathbf{n}) \cdot (Y - (Y \cdot \mathbf{n}) \cdot \mathbf{n}) &= 0 \end{aligned} \quad (4.33)$$

$$\iff (W \cdot \mathbf{n}) \cdot (Y \cdot \mathbf{n}) = W \cdot Y. \quad (4.34)$$

Now we want to introduce the concept of rotated vector fields over surfaces.

Given a vector field  $W$  over  $\mathbf{x}$ , a vector field  $W^{[\gamma]}$  is called *rotated by the angle  $\gamma$*  if the angle between the projections of  $W^{[\gamma]}$  and  $W$  into the tangent plane is  $\gamma$  for every point of the surface. Figure 4.4 illustrates the concepts of rotated and perpendicular vector fields.

Now we want to discuss how the curvature of a rotated vector field  $W^{[\gamma]}$  depends on  $\gamma$ .

Let  $W^{[\gamma]}$  be a rotated vector field of the vector field  $W = W^{[0]}$ . Then we obtain for its geodesic curvature  $\kappa_g^{[\gamma]}$ :

$$\kappa_g^{[\gamma]} = \kappa_g^{[0]} \cdot \cos \gamma + \kappa_g^{[\frac{\pi}{2}]} \cdot \sin \gamma. \quad (4.35)$$

For the curvature  $\kappa^{[\gamma]}$  we know from (4.32) :

$$\kappa^{[\gamma]} = \sqrt{(\kappa_n^{[\gamma]})^2 + (\kappa_g^{[\gamma]})^2} \quad (4.36)$$

where  $\kappa_n^{[\gamma]}$  denotes the normal curvature of the surface in the direction  $W_p^{[\gamma]}$ . Equation (4.35) informs us about the behavior of  $\kappa_g^{[\gamma]}$  while changing  $\gamma$ . The behavior of  $\kappa_n^{[\gamma]}$  is determined by Euler's theorem:

Let  $W_p^{[\gamma_0]}$  be one direction of extreme normal curvature of  $\mathbf{x}$ , i.e., one of the principal directions. Then we obtain:

$$\kappa_n^{[\gamma]} = \kappa_n^{[\gamma_0]} \cdot \cos^2(\gamma - \gamma_0) + \kappa_n^{[\gamma_0 + \frac{\pi}{2}]} \cdot \sin^2(\gamma - \gamma_0). \quad (4.37)$$

## 4.5 "Thickness" of Tangent Curves

We consider the special case that a scalar field  $s(u, v)$  over  $\mathbf{x}$  is given and the desired tangent curves are the equipotential lines of  $s$  on the surface. For

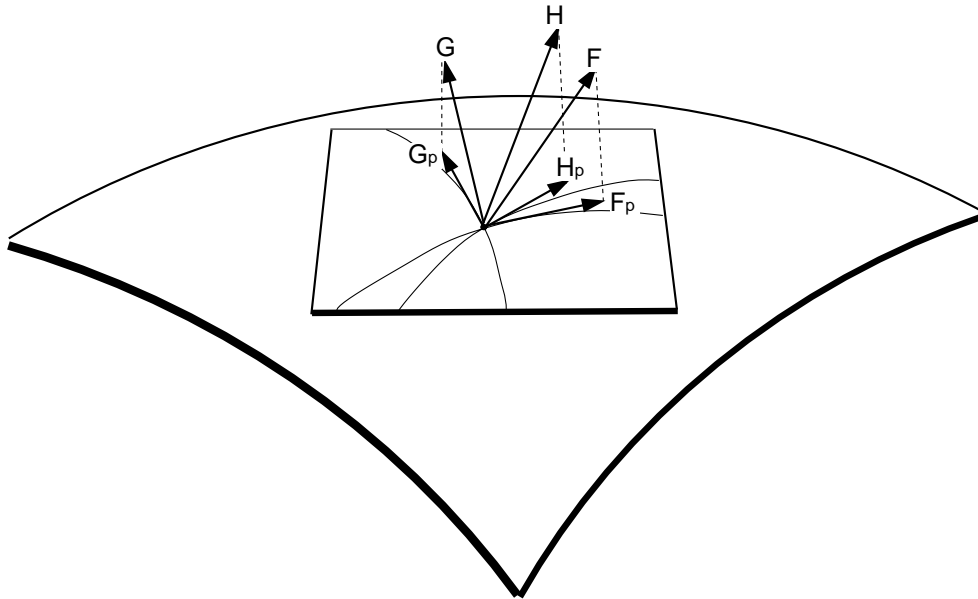


Figure 4.4: Rotated vector fields over a surface. Shown are the representatives of the vector fields  $W$ ,  $Y$  and  $Z$  for one point  $\mathbf{x}(u, v)$ , the tangent plane in  $\mathbf{x}(u, v)$  and the projections  $W_p$ ,  $Y_p$  and  $Z_p$  of  $W$ ,  $Y$  and  $Z$  into the tangent plane. Let us assume that the angle between  $W_p$  and  $Z_p$  is  $\frac{\pi}{6}$  and the angle between  $W_p$  and  $Y_p$  is  $\frac{\pi}{2}$ . Then we obtain  $W = W^{[0]}$ ,  $Y = W^{[\frac{\pi}{2}]}$  and  $Z = W^{[\frac{\pi}{6}]}$ .

this special case there is an easy way of drawing a "representative" of these curves: mark all those points on the surface which have a value of  $s$  in a fixed (small) interval. (The upper left pictures of the figures 6.2, 6.5, 6.6, 6.7, 6.8 and the lower left pictures of the figures 6.6 and 6.8 are generated this way).

The resulting "lines" on the surface are actually point sets with a changing "thickness". This "thickness" may provide information about the behavior of the surface and the tangent curves.

The "thickness" of a tangent curve on a surface  $\mathbf{x}(u, v)$  denotes how "strongly" the values of the scalar field change around  $\mathbf{x}(u, v)$ . A thin tangent curve indicates a strong change of the values of  $s$  around  $\mathbf{x}(u, v)$ .

We consider the scalar field  $s$  in the domain of the surface. A measure of how much  $s$  changes around a point  $(u, v)$  is the magnitude of the gradient

$(s_u, s_v)^T$ . We only have to map this gradient vector onto the surface in an appropriate form. The magnitude of the resulting vector on the surface tells us how much  $s$  changes on the surface around the point  $\mathbf{x}(u, v)$ .

The projection  $\mathbf{x}_{gr}$  of the gradient vector (or to be exactly: a vector perpendicular to the gradient vector but with the same magnitude) has the following equation:

$$\mathbf{x}_{gr} = \frac{-s_v \cdot \mathbf{x}_u + s_u \cdot \mathbf{x}_v}{\|\mathbf{x}_u \times \mathbf{x}_v\|}. \quad (4.38)$$

Then the "thickness"  $th(u, v)$  of the tangent curve through  $\mathbf{x}(u, v)$  can be expressed in the form:

$$th = \frac{1}{\|\mathbf{x}_{gr}\|} \quad (4.39)$$

(4.38) and (4.39) show that  $th$  around a critical point tends to infinity.

## 4.6 Geometric Continuity of Tangent Curves on Surfaces

Since we are able to compute the curvature of tangent curves it makes sense to ask for conditions for the surface to achieve a  $G^2$  continuity of the tangent curves.

We want to use the following definition for geometric continuity (see [4], [25]):

Two curves are  $G^r$  at a common point  $\mathbf{x}$  iff there exists a regular parametrization with respect to which they are  $C^r$  at  $\mathbf{x}$ . Two surfaces are  $G^r$  along a common line  $l$  iff there exists a regular parametrization with respect to which they are  $C^r$  along  $l$ .

It has been recognized that there are equivalent definitions for  $r = 1, 2$  which use geometric properties of the curve/surface:

Two curves through the point  $\mathbf{x}_0$  are  $G^2$  in this point iff

- the normalized tangent vectors coincide in  $\mathbf{x}_0$  and
- the osculating planes coincide in  $\mathbf{x}_0$  and
- the signed curvatures coincide in  $\mathbf{x}_0$ .

Two surfaces sharing a common line  $l$  are  $G^1$  along  $l$  iff their normalized

normal vectors coincide along  $l$ .

Two surfaces are  $G^2$  along  $l$  iff

- the normalized normal vectors coincide along  $l$  and
- the Dupin's indicatrices coincide along  $l$ .

In [22] there are some more geometric conditions for  $G^2$  surfaces.

In the next chapter we want to develop surface conditions for  $G^2$  continuity of particular tangent curves. Doing this we obtain some geometric conditions (necessary and sufficient) for  $G^3$  continuity of surfaces. Those conditions are formulated in the theorems 5 and 6.

## Chapter 5

# Particular Tangent Curves on Surfaces

In this chapter we want to apply the theoretical results from the previous chapter to concrete families of curves on surfaces. These curves are: contour lines, lines of curvature, asymptotic lines, isophotes and reflection lines.

All these curves have something in common:

- They reflect geometric properties of the surface, i.e. they do not depend on the parametrization of the surface.
- For sufficiently complicated surfaces (for instance bicubic polynomial surfaces), these lines can be described only as the solution of differential equations. The treatment of the curves themselves requires the numerical solution of those equations.

For all these curves, we want to compute their curvature and (if possible) their "thickness". Furthermore, we develop conditions for critical points and we look for conditions of  $G^2$ -continuity of these curves.

For computing the curvature of these curves we only have to show how the domain vector field  $V$  and its partial derivatives are computed. Then we can apply the results from section 4.2 to compute curvature and geodesic curvature.

## 5.1 Contour Lines

A family of contour lines is defined by a normalized direction vector  $\mathbf{r}$  in the 3D space. We consider all planes perpendicular to  $\mathbf{r}$ . The intersections of these planes with the surface yield a family of curves on the surface - the contour lines. Therefore, points on the surface are located on the same contour line referring to  $\mathbf{r}$  if the scalar field

$$s(u, v) = \mathbf{r} \cdot (\mathbf{x} - (0, 0, 0)^T) \quad (5.1)$$

gives the same values for those points. Thus, contour lines are the equipotential lines of the scalar field  $s$ . The direction of these lines in the domain can be computed as the perpendiculars to the gradients of  $s$ . Since the gradient of  $s$  is given by  $(s_u, s_v)^T$ , we obtain for the directions of the contour lines in the domain:

$$V = \begin{pmatrix} -s_v \\ s_u \end{pmatrix} = \begin{pmatrix} -\mathbf{r} \cdot \mathbf{x}_v \\ \mathbf{r} \cdot \mathbf{x}_u \end{pmatrix}. \quad (5.2)$$

From (5.2) we obtain

$$V_u = \begin{pmatrix} -s_{uv} \\ s_{uu} \end{pmatrix} = \begin{pmatrix} -\mathbf{r} \cdot \mathbf{x}_{uv} \\ \mathbf{r} \cdot \mathbf{x}_{uu} \end{pmatrix} \quad (5.3)$$

$$V_v = \begin{pmatrix} -s_{vv} \\ s_{uv} \end{pmatrix} = \begin{pmatrix} -\mathbf{r} \cdot \mathbf{x}_{vv} \\ \mathbf{r} \cdot \mathbf{x}_{uv} \end{pmatrix}. \quad (5.4)$$

**Remark 1:** If we take

$$V = \frac{1}{\|\mathbf{x}_u \times \mathbf{x}_v\|} \cdot \begin{pmatrix} -\mathbf{r} \cdot \mathbf{x}_v \\ \mathbf{r} \cdot \mathbf{x}_u \end{pmatrix} \quad (5.5)$$

we obtain the following formulas for the first and second derivative vectors of the surface curve:

$$\dot{\mathbf{x}} = \mathbf{n} \times \mathbf{r} \quad (5.6)$$

$$\ddot{\mathbf{x}} = \frac{(\mathbf{r} \cdot \mathbf{x}_u) \cdot (\mathbf{n}_v \times \mathbf{r}) - (\mathbf{r} \cdot \mathbf{x}_v) \cdot (\mathbf{n}_u \times \mathbf{r})}{\|\mathbf{x}_u \times \mathbf{x}_v\|}. \quad (5.7)$$

**Critical points:**

To obtain a critical point we have to have  $V = (0, 0)^T$ . Since we assume a



regularly parametrized surface, this occurs iff  $\mathbf{n}$  and  $\mathbf{r}$  have the same direction.

**Continuity:**

All contour lines through a point  $\mathbf{x}_0$  on  $\mathbf{x}$  are  $G^2$  iff  $\mathbf{x}$  is  $G^2$  in a neighborhood of  $\mathbf{x}_0$  (see [1]).

**”Thickness”:**

Since we know the scalar field  $s$ , we can use (4.38) and (4.39) to compute the ”thickness” of the contour lines. For contour lines, (4.39) can also be written in the form

$$th = \frac{1}{\|\mathbf{n} \times \mathbf{r}\|}. \quad (5.8)$$

## 5.2 Lines of Curvature

Lines of curvature are the tangent curves of the principal directions – considered as a vector field on the surface. Since there are two principal direction vector fields (whose vectors are perpendicular to each other) we have two families of lines of curvature.

The principal directions are the solutions  $(vx, vy)^T$  of the quadratic equation (see [4]):

$$\det \begin{bmatrix} vy^2 & -vx \cdot vy & vx^2 \\ E & F & G \\ L & M & N \end{bmatrix} = 0. \quad (5.9)$$

(5.9) yields two solution classes of  $(vx, vy)^T$  (where a solution class contains only vectors of the same direction). We use the abbreviations  $ha, hb$  and  $hc$  which are defined as:

$$\begin{pmatrix} ha \\ hb \\ hc \end{pmatrix} = \begin{pmatrix} E \\ F \\ G \end{pmatrix} \times \begin{pmatrix} L \\ M \\ N \end{pmatrix}. \quad (5.10)$$

This gives for the partial derivatives:

$$\begin{pmatrix} ha_u \\ hb_u \\ hc_u \end{pmatrix} = \begin{pmatrix} E_u \\ F_u \\ G_u \end{pmatrix} \times \begin{pmatrix} L \\ M \\ N \end{pmatrix} + \begin{pmatrix} E \\ F \\ G \end{pmatrix} \times \begin{pmatrix} L_u \\ M_u \\ N_u \end{pmatrix} \quad (5.11)$$

$$\begin{pmatrix} ha_v \\ hb_v \\ hc_v \end{pmatrix} = \begin{pmatrix} E_v \\ F_v \\ G_v \end{pmatrix} \times \begin{pmatrix} L \\ M \\ N \end{pmatrix} + \begin{pmatrix} E \\ F \\ G \end{pmatrix} \times \begin{pmatrix} L_v \\ M_v \\ N_v \end{pmatrix}. \quad (5.12)$$

Furthermore, we use the abbreviations

$$hd = hb^2 - 4 \cdot ha \cdot hc \quad (5.13)$$

$$hd_u = 2 \cdot hb \cdot hb_u - 4 \cdot ha_u \cdot hc - 4 \cdot ha \cdot hc_u \quad (5.14)$$

$$hd_v = 2 \cdot hb \cdot hb_v - 4 \cdot ha_v \cdot hc - 4 \cdot ha \cdot hc_v, \quad (5.15)$$

Then we can write two representatives of the solution classes of (5.9) – one for each class – in the form:

$$V_1 = \begin{pmatrix} vx_1 \\ vy_1 \end{pmatrix} = \begin{pmatrix} -2 \cdot ha + hb - \sqrt{hd} \\ 2 \cdot hc - hb - \sqrt{hd} \end{pmatrix} \quad (5.16)$$

$$V_2 = \begin{pmatrix} vx_2 \\ vy_2 \end{pmatrix} = \begin{pmatrix} -2 \cdot ha + hb + \sqrt{hd} \\ 2 \cdot hc - hb + \sqrt{hd} \end{pmatrix}. \quad (5.17)$$

This yields for the partial derivatives:

$$V_{1u} = \begin{pmatrix} -2 \cdot ha_u + hb_u - \frac{hd_u}{2 \cdot \sqrt{hd}} \\ 2 \cdot hc_u - hb_u - \frac{hd_u}{2 \cdot \sqrt{hd}} \end{pmatrix} \quad (5.18)$$

$$V_{1v} = \begin{pmatrix} -2 \cdot ha_v + hb_v - \frac{hd_v}{2 \cdot \sqrt{hd}} \\ 2 \cdot hc_v - hb_v - \frac{hd_v}{2 \cdot \sqrt{hd}} \end{pmatrix} \quad (5.19)$$

$$V_{2u} = \begin{pmatrix} -2 \cdot ha_u + hb_u + \frac{hd_u}{2 \cdot \sqrt{hd}} \\ 2 \cdot hc_u - hb_u + \frac{hd_u}{2 \cdot \sqrt{hd}} \end{pmatrix} \quad (5.20)$$

$$V_{2v} = \begin{pmatrix} -2 \cdot ha_v + hb_v + \frac{hd_v}{2 \cdot \sqrt{hd}} \\ 2 \cdot hc_v - hb_v + \frac{hd_v}{2 \cdot \sqrt{hd}} \end{pmatrix}. \quad (5.21)$$

**Critical points:**

occur iff  $V_1 = (0, 0)^T$  and  $V_2 = (0, 0)^T$ . Since

$$\begin{aligned} (vx_1 = 0) \wedge (vx_2 = 0) &\iff (hd = 0) \wedge (2 \cdot ha = hb) \\ (vy_1 = 0) \wedge (vy_2 = 0) &\iff (hd = 0) \wedge (2 \cdot hc = hb), \end{aligned}$$

this is only possible for  $2 \cdot ha = hb = 2 \cdot hc$ . This and (5.10) give  $0 = (ha, 2 \cdot ha, ha)^T \cdot (E, F, G)^T$ . Since  $\mathbf{x}$  is regularly parametrized, this is only possible for  $ha = 0 = hb = hc$ , i.e. we have an umbilical point. Therefore, lines of curvature produce critical points in (and only in) umbilical points on the surface.

**Continuity:**

The correlation between the geometric continuities of the lines of curvature and the surface is given by

**Theorem 5** *Given are two surfaces  $\mathbf{x}$  and  $\tilde{\mathbf{x}}$  which join along a common line  $l$ . Furthermore, every point on  $l$  is non-umbilical in  $\mathbf{x}$  and  $\tilde{\mathbf{x}}$ , and in no point of  $l$  the lines of curvature of  $\mathbf{x}$  and  $\tilde{\mathbf{x}}$  are tangent to  $l$ . Then  $\mathbf{x}$  and  $\tilde{\mathbf{x}}$  are  $G^3$  along  $l$  iff their lines of curvature are  $G^2$  across  $l$ .*

**Proof:**

"  $\Rightarrow$  ": If  $\mathbf{x}$  and  $\tilde{\mathbf{x}}$  are  $G^3$  along  $l$  they can be reparametrized in a way that they coincide in all partial derivatives of order  $\leq 3$ . Since the curvature formula of the lines of curvature contains only those derivatives (see above), the lines of curvature are  $G^2$ .

"  $\Leftarrow$  ": We assume that the junction line  $l$  is  $(0, v), 0 \leq v \leq 1$ . This can be done by a linear reparametrization of  $\mathbf{x}$  and  $\tilde{\mathbf{x}}$  without loss of generality.

The  $G^2$  condition of the lines of curvature contains coincidence in surface normal, principal directions and principal curvatures, therefore  $G^2$  of the surfaces along  $l$ . Thus, we can assume that  $\mathbf{x}$  and  $\tilde{\mathbf{x}}$  are parametrized in a way that

$$\begin{aligned} \mathbf{x}(0, v) &= \tilde{\mathbf{x}}(0, v) \quad ; \quad \mathbf{x}_u(0, v) = \tilde{\mathbf{x}}_u(0, v) \\ \mathbf{x}_v(0, v) &= \tilde{\mathbf{x}}_v(0, v) \quad ; \quad \mathbf{x}_{uu}(0, v) = \tilde{\mathbf{x}}_{uu}(0, v) \\ \mathbf{x}_{uv}(0, v) &= \tilde{\mathbf{x}}_{uv}(0, v) \quad ; \quad \mathbf{x}_{vv}(0, v) = \tilde{\mathbf{x}}_{vv}(0, v). \end{aligned} \tag{5.22}$$

From (5.22) we obtain

$$\begin{aligned}\mathbf{x}_{uvw}(0, v) &= \tilde{\mathbf{x}}_{uvw}(0, v) \\ \mathbf{x}_{uvv}(0, v) &= \tilde{\mathbf{x}}_{uvv}(0, v) \\ \mathbf{x}_{vvv}(0, v) &= \tilde{\mathbf{x}}_{vvv}(0, v).\end{aligned}\quad (5.23)$$

Let  $\dot{\mathbf{x}}_1$  and  $\dot{\mathbf{x}}_2$  be the tangent vectors of the lines of curvature on  $\mathbf{x}$ . Furthermore, let  $\dot{\tilde{\mathbf{x}}}_1$  and  $\dot{\tilde{\mathbf{x}}}_2$  be the tangent vectors of the lines of curvature on  $\tilde{\mathbf{x}}$ . Then (4.21), (5.16), (5.17) and (5.22) give

$$\dot{\mathbf{x}}_1(0, v) = (vx_1 \cdot \mathbf{x}_u + vy_1 \cdot \mathbf{x}_v)(0, v) = \dot{\tilde{\mathbf{x}}}_1(0, v) \quad (5.24)$$

$$\dot{\mathbf{x}}_2(0, v) = (vx_2 \cdot \mathbf{x}_u + vy_2 \cdot \mathbf{x}_v)(0, v) = \dot{\tilde{\mathbf{x}}}_2(0, v) \quad (5.25)$$

where  $vx_1, vy_1, vx_2, vy_2$  are given by (5.16) and (5.17).

Let  $\ddot{\mathbf{x}}_1$  and  $\ddot{\mathbf{x}}_2$  be the second derivative vectors of the lines of curvature of  $\mathbf{x}$ , and let  $\ddot{\tilde{\mathbf{x}}}_1$  and  $\ddot{\tilde{\mathbf{x}}}_2$  be the second derivative vectors of the lines of curvature of  $\tilde{\mathbf{x}}$ . Then (4.23), (5.16) – (5.21) and (5.22) give

$$\ddot{\tilde{\mathbf{x}}}_1(0, v) - \ddot{\mathbf{x}}_1(0, v) = vx_1 \cdot (\mathbf{n} \cdot (\tilde{\mathbf{x}}_{uuu} - \mathbf{x}_{uuu})) \cdot (a_1 \cdot \mathbf{x}_u + b_1 \cdot \mathbf{x}_v) \quad (5.26)$$

$$\ddot{\tilde{\mathbf{x}}}_2(0, v) - \ddot{\mathbf{x}}_2(0, v) = vx_2 \cdot (\mathbf{n} \cdot (\tilde{\mathbf{x}}_{uuu} - \mathbf{x}_{uuu})) \cdot (a_2 \cdot \mathbf{x}_u + b_2 \cdot \mathbf{x}_v) \quad (5.27)$$

where

$$a_1 = \frac{2 \cdot ha \cdot F + hb \cdot G}{\sqrt{hd}} - G \quad (5.28)$$

$$b_1 = \frac{2 \cdot ha \cdot F + hb \cdot G}{\sqrt{hd}} + 2 \cdot F + G \quad (5.29)$$

$$a_2 = \frac{2 \cdot ha \cdot F + hb \cdot G}{\sqrt{hd}} + G \quad (5.30)$$

$$b_2 = \frac{2 \cdot ha \cdot F + hb \cdot G}{\sqrt{hd}} - 2 \cdot F - G. \quad (5.31)$$

(The assumption that no umbilical point is on the junction line  $l$  ensures that  $hd > 0$  along  $l$ .)

Now the  $G^2$  condition of the lines of curvature across  $l$  can be formulated in the following way:

$$(\ddot{\tilde{\mathbf{x}}}_1(0, v) - \ddot{\mathbf{x}}_1(0, v)) \quad \text{parallel to} \quad \dot{\mathbf{x}}_1(0, v) \quad (5.32)$$

$$(\ddot{\tilde{\mathbf{x}}}_2(0, v) - \ddot{\mathbf{x}}_2(0, v)) \quad \text{parallel to} \quad \dot{\mathbf{x}}_2(0, v). \quad (5.33)$$

Using (5.24), (5.25), (5.26), (5.27) and the fact that  $\mathbf{x}_u$  and  $\mathbf{x}_v$  are linearly independent, we can write (5.32) and (5.33) in the form

$$vx_1 \cdot (\mathbf{n} \cdot (\tilde{\mathbf{x}}_{uuu} - \mathbf{x}_{uuu})) \cdot \det_1 = 0 \quad (5.34)$$

$$vx_2 \cdot (\mathbf{n} \cdot (\tilde{\mathbf{x}}_{uuu} - \mathbf{x}_{uuu})) \cdot \det_2 = 0 \quad (5.35)$$

where

$$\det_1 = \det \begin{bmatrix} vx_1 & a_1 \\ vy_1 & b_1 \end{bmatrix} \quad (5.36)$$

$$\det_2 = \det \begin{bmatrix} vx_2 & a_2 \\ vy_2 & b_2 \end{bmatrix}. \quad (5.37)$$

(5.24), (5.25) and the assumption that the lines of curvature are not parallel to  $l$  give

$$vx_1 \cdot vx_2 \neq 0. \quad (5.38)$$

From (5.28) - (5.31), (5.36) and (5.37) we obtain

$$\det_1 \cdot \det_2 = \frac{vx_1^2 \cdot vx_2^2 \cdot (F^2 - E \cdot G)}{hd}. \quad (5.39)$$

This, (5.38) and the assumption that  $\mathbf{x}$  and  $\tilde{\mathbf{x}}$  are regularly parametrized yield

$$\det_1 \cdot \det_2 \neq 0. \quad (5.40)$$

From (5.34), (5.35), (5.38) and (5.40) we obtain

$$(\mathbf{n} \cdot (\tilde{\mathbf{x}}_{uuu} - \mathbf{x}_{uuu}))(0, v) = 0. \quad (5.41)$$

Because of (5.41), there exist two scalar functions  $r_1(v)$  and  $r_2(v)$  so that

$$\tilde{\mathbf{x}}_{uuu}(0, v) = \mathbf{x}_{uuu}(0, v) + r_1(v) \cdot \mathbf{x}_u(0, v) + r_2(v) \cdot \mathbf{x}_v(0, v). \quad (5.42)$$

Now we look for a reparametrization  $\hat{\mathbf{x}}$  of  $\mathbf{x}$  which is  $C^3$  to  $\tilde{\mathbf{x}}$  along  $l$ . We define

$$\hat{\mathbf{x}}(u, v) := \mathbf{x}(\hat{u}(u, v), \hat{v}(u, v)) \quad (5.43)$$

where

$$\hat{u}(u, v) = u + \frac{1}{6} \cdot u^3 \cdot r_1(v) \quad (5.44)$$

$$\hat{v}(u, v) = v + \frac{1}{6} \cdot u^3 \cdot r_2(v). \quad (5.45)$$

Considering (5.44) and (5.45) to the junction line  $l$  (i.e., setting  $u = 0$ ), we obtain:

$$\hat{u}(0, v) = 0 \ ; \ \hat{u}_u(0, v) = 1 \ ; \ \hat{u}_{uu}(0, v) = 0 \ ; \ \hat{u}_{uuu}(0, v) = r_1(v) \quad (5.46)$$

$$\hat{v}(0, v) = v \ ; \ \hat{v}_u(0, v) = 0 \ ; \ \hat{v}_{uu}(0, v) = 0 \ ; \ \hat{v}_{uuu}(0, v) = r_2(v). \quad (5.47)$$

Applying the chain rule to (5.43), we obtain for the  $u$ -partials of  $\hat{\mathbf{x}}$ :

$$\hat{\mathbf{x}}_u = \hat{u}_u \cdot \mathbf{x}_u + \hat{v}_u \cdot \mathbf{x}_v \quad (5.48)$$

$$\begin{aligned} \hat{\mathbf{x}}_{uu} &= \hat{u}_u^2 \cdot \mathbf{x}_{uu} + 2 \cdot \hat{u}_u \cdot \hat{v}_u \cdot \mathbf{x}_{uv} + \hat{v}_u^2 \cdot \mathbf{x}_{vv} \\ &\quad + \hat{u}_{uu} \cdot \mathbf{x}_u + \hat{v}_{uu} \cdot \mathbf{x}_v \end{aligned} \quad (5.49)$$

$$\begin{aligned} \hat{\mathbf{x}}_{uuu} &= \hat{u}_u^3 \cdot \mathbf{x}_{uuu} + 3 \cdot \hat{u}_u^2 \cdot \hat{v}_u \cdot \mathbf{x}_{uuv} + 3 \cdot \hat{u}_u \cdot \hat{v}_u^2 \cdot \mathbf{x}_{uvv} + \hat{v}_u^3 \cdot \mathbf{x}_{vvv} \\ &\quad + 3 \cdot (\hat{u}_u \cdot \hat{u}_{uu} \cdot \mathbf{x}_{uu} + (\hat{v}_u \cdot \hat{u}_{uu} + \hat{u}_u \cdot \hat{v}_{uu}) \cdot \mathbf{x}_{uv} + \hat{v}_u \cdot \hat{v}_{uu} \cdot \mathbf{x}_{vv}) \\ &\quad + \hat{u}_{uuu} \cdot \mathbf{x}_u + \hat{v}_{uuu} \cdot \mathbf{x}_v. \end{aligned} \quad (5.50)$$

Setting  $u = 0$ , we obtain from (5.48) - (5.50) using (5.46) and (5.47):

$$\hat{\mathbf{x}}(0, v) = \mathbf{x}(0, v) = \tilde{\mathbf{x}}(0, v) \quad (5.51)$$

$$\hat{\mathbf{x}}_u(0, v) = \mathbf{x}_u(0, v) = \tilde{\mathbf{x}}_u(0, v) \quad (5.52)$$

$$\hat{\mathbf{x}}_{uu}(0, v) = \mathbf{x}_{uu}(0, v) = \tilde{\mathbf{x}}_{uu}(0, v) \quad (5.53)$$

$$\begin{aligned} \hat{\mathbf{x}}_{uuu}(0, v) &= \mathbf{x}_{uuu}(0, v) + r_1(v) \cdot \mathbf{x}_u(0, v) + r_2(v) \cdot \mathbf{x}_v(0, v) \\ &= \tilde{\mathbf{x}}_{uuu}(0, v). \end{aligned} \quad (5.54)$$

From (5.51) - (5.54) we obtain

$$\begin{aligned} \hat{\mathbf{x}}_v(0, v) &= \tilde{\mathbf{x}}_v(0, v) \ ; \ \hat{\mathbf{x}}_{uv}(0, v) = \tilde{\mathbf{x}}_{uv}(0, v) \\ \hat{\mathbf{x}}_{vv}(0, v) &= \tilde{\mathbf{x}}_{vv}(0, v) \ ; \ \hat{\mathbf{x}}_{uuv}(0, v) = \tilde{\mathbf{x}}_{uuv}(0, v) \\ \hat{\mathbf{x}}_{uvv}(0, v) &= \tilde{\mathbf{x}}_{uvv}(0, v) \ ; \ \hat{\mathbf{x}}_{vvv}(0, v) = \tilde{\mathbf{x}}_{vvv}(0, v). \end{aligned} \quad (5.55)$$

Therefore,  $\hat{\mathbf{x}}$  and  $\tilde{\mathbf{x}}$  are  $C^3$  along  $l$ , which gives that  $\mathbf{x}$  and  $\tilde{\mathbf{x}}$  are  $G^3$  along  $l$ .  
□

**Remark 2:** If there is only a single point  $\mathbf{x}_s$  on the junction line  $l$  which is umbilic or in which one of the lines of curvature is tangent to  $l$ , this point  $\mathbf{x}_s$  divides  $l$  in two parts which both (except for  $\mathbf{x}_s$  itself) fulfill theorem 5. Since  $\mathbf{x}$  and  $\tilde{\mathbf{x}}$  are continuous, we still can infer  $G^3$  of the surface from  $G^2$  of the lines of curvature across  $l \setminus \{\mathbf{x}_s\}$ .

If  $l$  and a line of curvature coincide, we have to demand  $G^3$  of the other line of curvature across  $l$  for obtaining a  $G^3$  surface.

**Remark 3:** The proof of theorem 5 used the assumption that *both* lines of curvature are  $G^2$  across  $l$  only for making sure that  $\mathbf{x}$  and  $\tilde{\mathbf{x}}$  are  $G^2$  along  $l$ . Therefore, we can rewrite theorem 5 in the following form:

*Given are two surfaces  $\mathbf{x}$  and  $\tilde{\mathbf{x}}$  which are  $G^2$  along a common line  $l$ . Furthermore, every point on  $l$  is non-umbilical in  $\mathbf{x}$  and  $\tilde{\mathbf{x}}$ , and in no point of  $l$  the lines of curvature of  $\mathbf{x}$  and  $\tilde{\mathbf{x}}$  are tangent or perpendicular to  $l$ . Then  $\mathbf{x}$  and  $\tilde{\mathbf{x}}$  are  $G^3$  along  $l$  iff there is one family of lines of curvature which is  $G^2$  across  $l$ .*

**Remark 4:** Theorem 5 has some similarities to the *linkage curve theorem* described in [22]. In this theorem, the sufficient condition for  $G^2$  of two surfaces along a common line  $l$  is the continuity of the normal curvature of a family of surface curves across  $l$ . That means, both theorem 5 and the linkage curve theorem use curvature properties of families of curves across the junction line to obtain conditions for geometric continuity of the surface.

## 5.3 Asymptotic Lines

Asymptotic lines are defined by the vector field  $(vx, vy)^T$  that satisfies (see [4]):

$$L \cdot vx^2 + 2 \cdot M \cdot vx \cdot vy + N \cdot vy^2 = 0. \quad (5.56)$$

We have two real solution classes for negative Gaussian curvature, one solution class for zero Gaussian curvature and only complex solutions for positive Gaussian curvature (see [4]). For negative Gaussian curvature, the directions of the asymptotic lines of the surface coincide with the directions of the Dupin's indicatrices (in this case a pair of hyperbolas). The defining geometric property of asymptotic lines is a zero normal curvature in every point of the surface. Here we only consider the case of negative Gaussian curvature. Using the abbreviations

$$he = M^2 - L \cdot N \quad (5.57)$$

$$he_u = 2 \cdot M \cdot M_u - L_u \cdot N - L \cdot N_u \quad (5.58)$$

$$he_v = 2 \cdot M \cdot M_v - L_v \cdot N - L \cdot N_v, \quad (5.59)$$

we can write two representatives of the solution classes in the following form:

$$V_1 = \begin{pmatrix} N - M - \sqrt{he} \\ L - M + \sqrt{he} \end{pmatrix} \quad (5.60)$$

$$V_2 = \begin{pmatrix} N - M + \sqrt{he} \\ L - M - \sqrt{he} \end{pmatrix}. \quad (5.61)$$

This yields for the partial derivatives:

$$V_{1u} = \begin{pmatrix} N_u - M_u - \frac{he_u}{2\sqrt{he}} \\ L_u - M_u + \frac{he_u}{2\sqrt{he}} \end{pmatrix} \quad (5.62)$$

$$V_{1v} = \begin{pmatrix} N_v - M_v - \frac{he_v}{2\sqrt{he}} \\ L_v - M_v + \frac{he_v}{2\sqrt{he}} \end{pmatrix} \quad (5.63)$$

$$V_{2u} = \begin{pmatrix} N_u - M_u + \frac{he_u}{2\sqrt{he}} \\ L_u - M_u - \frac{he_u}{2\sqrt{he}} \end{pmatrix} \quad (5.64)$$

$$V_{2v} = \begin{pmatrix} N_v - M_v + \frac{he_v}{2\sqrt{he}} \\ L_v - M_v - \frac{he_v}{2\sqrt{he}} \end{pmatrix}. \quad (5.65)$$

**critical Points:**

occur iff  $V_1 = (0, 0)^T$  and  $V_2 = (0, 0)^T$ . Since

$$\begin{aligned} (vx_1 = 0) \wedge (vx_2 = 0) &\iff (he = 0) \wedge (N = M) \\ (vy_1 = 0) \wedge (vy_2 = 0) &\iff (he = 0) \wedge (L = M), \end{aligned}$$

this is only possible for  $L = M = N$ . This gives a zero Gaussian curvature. Therefore, in the considered areas of negative Gaussian curvature there are no critical points.

**Continuity:**

**Theorem 6** *Given are two surfaces  $\mathbf{x}$  and  $\tilde{\mathbf{x}}$  which join along a common line  $l$ . Furthermore, every point on  $l$  has negative Gaussian curvature in  $\mathbf{x}$  and  $\tilde{\mathbf{x}}$ , and in no point of  $l$  one of the asymptotic lines of  $\mathbf{x}$  and  $\tilde{\mathbf{x}}$  is tangent to  $l$ . Then  $\mathbf{x}$  and  $\tilde{\mathbf{x}}$  are  $G^3$  along  $l$  iff their asymptotic lines are  $G^2$  across  $l$ .*



Most parts of the **proof** are similar to the proof of theorem 5.

" $\Rightarrow$ ": similar to theorem 5.

" $\Leftarrow$ ": We assume that the junction line  $l$  is  $(0, v), 0 \leq v \leq 1$ .

The  $G^2$  condition of the asymptotic lines gives  $G^2$  of the surfaces along  $l$  (see [22]). Thus, we can assume that  $\mathbf{x}$  and  $\tilde{\mathbf{x}}$  are parametrized in a way that

$$\begin{aligned} \mathbf{x}(0, v) &= \tilde{\mathbf{x}}(0, v) & ; & & \mathbf{x}_u(0, v) &= \tilde{\mathbf{x}}_u(0, v) \\ \mathbf{x}_v(0, v) &= \tilde{\mathbf{x}}_v(0, v) & ; & & \mathbf{x}_{uu}(0, v) &= \tilde{\mathbf{x}}_{uu}(0, v) \\ \mathbf{x}_{uv}(0, v) &= \tilde{\mathbf{x}}_{uv}(0, v) & ; & & \mathbf{x}_{vv}(0, v) &= \tilde{\mathbf{x}}_{vv}(0, v). \end{aligned} \quad (5.66)$$

From (5.66) we obtain

$$\begin{aligned} \mathbf{x}_{uvv}(0, v) &= \tilde{\mathbf{x}}_{uvv}(0, v) \\ \mathbf{x}_{vvv}(0, v) &= \tilde{\mathbf{x}}_{vvv}(0, v). \end{aligned} \quad (5.67)$$

Let  $\dot{\mathbf{x}}_1$  and  $\dot{\mathbf{x}}_2$  be the tangent vectors of the asymptotic lines on  $\mathbf{x}$ . Furthermore, let  $\dot{\tilde{\mathbf{x}}}_1$  and  $\dot{\tilde{\mathbf{x}}}_2$  be the tangent vectors of the asymptotic lines on  $\tilde{\mathbf{x}}$ . Then (4.21), (5.60), (5.61) and (5.66) give

$$\dot{\mathbf{x}}_1(0, v) = (vx_1 \cdot \mathbf{x}_u + vy_1 \cdot \mathbf{x}_v)(0, v) = \dot{\tilde{\mathbf{x}}}_1(0, v) \quad (5.68)$$

$$\dot{\mathbf{x}}_2(0, v) = (vx_2 \cdot \mathbf{x}_u + vy_2 \cdot \mathbf{x}_v)(0, v) = \dot{\tilde{\mathbf{x}}}_2(0, v) \quad (5.69)$$

where  $vx_1, vy_1, vx_2, vy_2$  are given by (5.60) and (5.61).

Let  $\ddot{\mathbf{x}}_1$  and  $\ddot{\mathbf{x}}_2$  be the second derivative vectors of the asymptotic lines of  $\mathbf{x}$ , and let  $\ddot{\tilde{\mathbf{x}}}_1$  and  $\ddot{\tilde{\mathbf{x}}}_2$  be the second derivative vectors of the asymptotic lines of  $\tilde{\mathbf{x}}$ . Then (4.23), (5.60) – (5.65) and (5.66) give

$$\ddot{\tilde{\mathbf{x}}}_1(0, v) - \ddot{\mathbf{x}}_1(0, v) = \frac{\mathbf{n} \cdot (\tilde{\mathbf{x}}_{uuu} - \mathbf{x}_{uuu})}{2 \cdot \sqrt{M^2 - L \cdot N}} \cdot (a_1 \cdot \mathbf{x}_u + b_1 \cdot \mathbf{x}_v) \quad (5.70)$$

$$\ddot{\tilde{\mathbf{x}}}_2(0, v) - \ddot{\mathbf{x}}_2(0, v) = -\frac{\mathbf{n} \cdot (\tilde{\mathbf{x}}_{uuu} - \mathbf{x}_{uuu})}{2 \cdot \sqrt{M^2 - L \cdot N}} \cdot (a_2 \cdot \mathbf{x}_u + b_2 \cdot \mathbf{x}_v) \quad (5.71)$$

where

$$a_1 = vx_1 \cdot N \quad (5.72)$$

$$b_1 = vx_1 \cdot (2 \cdot M - 3 \cdot N + 2 \cdot vx_2) \quad (5.73)$$

$$a_2 = vx_2 \cdot N \quad (5.74)$$

$$b_2 = vx_2 \cdot (2 \cdot M - 3 \cdot N + 2 \cdot vx_1). \quad (5.75)$$

(The assumption of negative Gaussian curvature along  $l$  ensures that  $M^2 - L \cdot N > 0$  along  $l$ .)

Now the  $G^2$  condition of the asymptotic lines across  $l$  can be formulated in the following way:

$$(\ddot{\tilde{\mathbf{x}}}_1(0, v) - \ddot{\mathbf{x}}_1(0, v)) \quad \text{parallel to} \quad \dot{\mathbf{x}}_1(0, v) \quad (5.76)$$

$$(\ddot{\tilde{\mathbf{x}}}_2(0, v) - \ddot{\mathbf{x}}_2(0, v)) \quad \text{parallel to} \quad \dot{\mathbf{x}}_2(0, v). \quad (5.77)$$

Using (5.68), (5.69), (5.70), (5.71) and the fact that  $\mathbf{x}_u$  and  $\mathbf{x}_v$  are linearly independent, we can write (5.76) and (5.77) in the form

$$(\mathbf{n} \cdot (\tilde{\mathbf{x}}_{uuu} - \mathbf{x}_{uuu})) \cdot \det_1 = 0 \quad (5.78)$$

$$(\mathbf{n} \cdot (\tilde{\mathbf{x}}_{uuu} - \mathbf{x}_{uuu})) \cdot \det_2 = 0 \quad (5.79)$$

where

$$\det_1 = \det \begin{bmatrix} vx_1 & a_1 \\ vy_1 & b_1 \end{bmatrix} \quad (5.80)$$

$$\det_2 = \det \begin{bmatrix} vx_2 & a_2 \\ vy_2 & b_2 \end{bmatrix}. \quad (5.81)$$

(5.68), (5.69) and the assumption that the lines of curvature are not parallel to  $l$  give

$$vx_1 \cdot vx_2 \neq 0. \quad (5.82)$$

From (5.72) - (5.75), (5.80) and (5.81) we obtain

$$\det_1 \cdot \det_2 = (vx_1 \cdot vx_2)^3. \quad (5.83)$$

This and (5.82) yield

$$\det_1 \cdot \det_2 \neq 0. \quad (5.84)$$

From (5.78), (5.79) and (5.84) we obtain

$$(\mathbf{n} \cdot (\tilde{\mathbf{x}}_{uuu} - \mathbf{x}_{uuu}))(0, v) = 0. \quad (5.85)$$

Continuing from (5.41), the rest of the proof is similar to the proof of theorem 5.  $\square$

## 5.4 Isophotes

Isophotes are first discussed as a surface interrogation tool in [23]. A family of isophotes is defined by an eye point  $\mathbf{e} = (ex, ey, ez)^T$ . Then the isophotes are the equipotential lines of the scalar field

$$s(u, v) = \frac{(\mathbf{e} - \mathbf{x}) \cdot \mathbf{n}}{\|\mathbf{e} - \mathbf{x}\|} \quad (5.86)$$

on the surface. That means, an isophote on a surface contains all surface points which have the same angle between the eye vector (i.e., eye point minus surface point) and the normal vector. Similar to section 5.1, we obtain for the domain vector field:

$$V = \begin{pmatrix} -s_v \\ s_u \end{pmatrix} \quad (5.87)$$

$$V_u = \begin{pmatrix} -s_{uv} \\ s_{uu} \end{pmatrix} \quad (5.88)$$

$$V_v = \begin{pmatrix} -s_{vv} \\ s_{uv} \end{pmatrix}. \quad (5.89)$$

### Critical points:

Given a point  $\mathbf{x}_0 = (x_0, y_0, z_0)^T = \mathbf{x}(u_0, v_0)$  on the surface, we want to find all eye points  $\mathbf{e}$  for which the corresponding isophotes have a critical point in  $\mathbf{x}_0$ . The critical point conditions obtained by this approach will give us information about the surface itself.

We assume that the surface is parametrized by the lines of curvature. Furthermore, we assume a parametrization by the arc length of the lines of curvature through  $\mathbf{x}_0$ . We want  $\mathbf{x}$  to be transformed such that

$$\mathbf{x}_0 = \mathbf{0} = (0, 0, 0)^T \quad (5.90)$$

$$\mathbf{x}_{0u} = \mathbf{x}_u(u_0, v_0) = (1, 0, 0)^T \quad (5.91)$$

$$\mathbf{x}_{0v} = \mathbf{x}_v(u_0, v_0) = (0, 1, 0)^T. \quad (5.92)$$

Then the assumption of parametrization by the lines of curvature yields

$$z_{0uv} = z_{uv}(u_0, v_0) = 0 \quad (5.93)$$

$$z_{0uu} = z_{uu}(u_0, v_0) = \kappa_1 \quad (5.94)$$

$$z_{0vv} = z_{vv}(u_0, v_0) = \kappa_2 \quad (5.95)$$

where  $\kappa_1$  and  $\kappa_2$  are the principal curvatures of  $\mathbf{x}$  in  $\mathbf{x}_0$ . (5.91) and (5.92) yield

$$\mathbf{n}_0 = \mathbf{n}(u_0, v_0) = (0, 0, 1)^T. \quad (5.96)$$

From (5.90) and (5.96) we obtain

$$\mathbf{n}_0 \cdot (\mathbf{e} - \mathbf{x}_0) = ez. \quad (5.97)$$

Applying (5.90) - (5.97) to (5.87), we obtain for  $V_0 = V(u_0, v_0)$ :

$$V_0 = \frac{1}{\|\mathbf{e} - \mathbf{x}_0\|^3} \cdot \begin{pmatrix} ey \cdot (\kappa_2 \cdot (\mathbf{e} - \mathbf{x}_0)^2 - \mathbf{n}_0 \cdot (\mathbf{e} - \mathbf{x}_0)) \\ -ex \cdot (\kappa_1 \cdot (\mathbf{e} - \mathbf{x}_0)^2 - \mathbf{n}_0 \cdot (\mathbf{e} - \mathbf{x}_0)) \end{pmatrix}. \quad (5.98)$$

We seek all points  $\mathbf{e}$  for which  $V_0 = (0, 0)^T$ . For that purpose we make a case distinction:

**case 1:**  $ex = 0$  and  $ey = 0$

which makes  $V_0 = (0, 0)^T$  obviously. It means that  $\mathbf{e}$  is on the normal line of  $\mathbf{x}_0$ .

**case 2:**  $ex = 0$  and  $ey \neq 0$ .

$ex = 0$  means that  $\mathbf{e}$  is in the plane through  $\mathbf{x}_0$  which contains  $\mathbf{n}$  and the principal direction  $\mathbf{x}_{0v}$ . We have to achieve

$$\kappa_2 \cdot (\mathbf{e} - \mathbf{x}_0)^2 = \mathbf{n}_0 \cdot (\mathbf{e} - \mathbf{x}_0). \quad (5.99)$$

Setting  $\alpha := \sphericalangle(\mathbf{n}, \mathbf{e} - \mathbf{x}_0)$ , we can write (5.99) in the form

$$\kappa_2 \cdot \|\mathbf{e} - \mathbf{x}_0\| = \cos \alpha. \quad (5.100)$$

Setting  $r_2 := 1/\kappa_2$ , (5.100) takes the form

$$\|\mathbf{e} - \mathbf{x}_0\| = r_2 \cdot \cos \alpha. \quad (5.101)$$

All (and only these) points of a circle with the center  $\mathbf{x}_0 + \frac{r_2}{2} \cdot \mathbf{n}_0$  and the radius  $\frac{r_2}{2}$  will satisfy (5.101), (see figure 5.1).

**case 3:**  $ex \neq 0$  and  $ey = 0$ :

analogous to case 2. As solution set for  $\mathbf{e}$  we obtain a circle in the plane containing  $\mathbf{n}_0$  and  $\mathbf{x}_{0u}$  with the center  $\mathbf{x}_0 + \frac{1}{2 \cdot \kappa_1} \cdot \mathbf{n}_0$  and the radius  $\frac{1}{2 \cdot \kappa_1}$ .

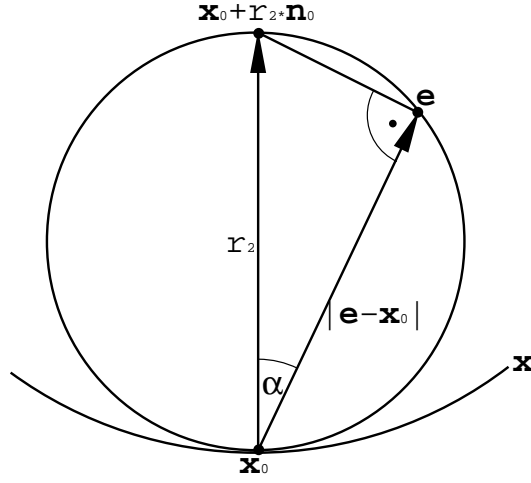


Figure 5.1: Cross section through  $\mathbf{x}_0$ , containing  $\mathbf{n}_0$  and  $\mathbf{x}_{0v}$ . All points  $\mathbf{e}$  on the circle produce a critical point in  $\mathbf{x}_0$ .

**case 4:**  $ex \neq 0$  and  $ey \neq 0$ .

In this case we have to achieve

$$\begin{aligned} \kappa_2 \cdot (\mathbf{e} - \mathbf{x}_0)^2 &= \mathbf{n}_0 \cdot (\mathbf{e} - \mathbf{x}_0) \\ \text{and } \kappa_1 \cdot (\mathbf{e} - \mathbf{x}_0)^2 &= \mathbf{n}_0 \cdot (\mathbf{e} - \mathbf{x}_0), \end{aligned} \quad (5.102)$$

which is only possible for  $\kappa_1 = \kappa_2$ , i.e.  $\mathbf{x}_0$  is an umbilical point. Then the solution set for  $\mathbf{e}$  is a sphere with the center  $\mathbf{x}_0 + \frac{1}{2 \cdot \kappa_1} \cdot \mathbf{n}_0 = \mathbf{x}_0 + \frac{1}{2 \cdot \kappa_2} \cdot \mathbf{n}_0$  and the radius  $\frac{1}{2 \cdot \kappa_1} = \frac{1}{2 \cdot \kappa_2}$ .

Figure 5.2 illustrates the solution of the critical point problem for isophotes. The planes  $p_1$  and  $p_2$  through  $\mathbf{x}_0$  contain  $\mathbf{n}_0$  and one principal direction in  $\mathbf{x}_0$ . The circle  $c_1$  in  $p_1$  has the center  $\mathbf{x}_0 + \frac{1}{2 \cdot \kappa_1} \cdot \mathbf{n}_0$  and the radius  $\frac{1}{2 \cdot \kappa_1}$ . The circle  $c_2$  in  $p_2$  has the center  $\mathbf{x}_0 + \frac{1}{2 \cdot \kappa_2} \cdot \mathbf{n}_0$  and the radius  $\frac{1}{2 \cdot \kappa_2}$ . ( $\kappa_1$  and  $\kappa_2$  denote the principal curvatures in  $\mathbf{x}_0$ ). The normal line of  $\mathbf{x}$  in  $\mathbf{x}_0$  is marked  $nl$ . Then the set of all eye points producing a critical point in isophotes is  $nl \cup c_1 \cup c_2$ .

**Remark 5:** For a negative Gaussian curvature in  $\mathbf{x}_0$  the two circles shown in figure 5.2 are located on different sides of the tangent plane in  $\mathbf{x}_0$ .

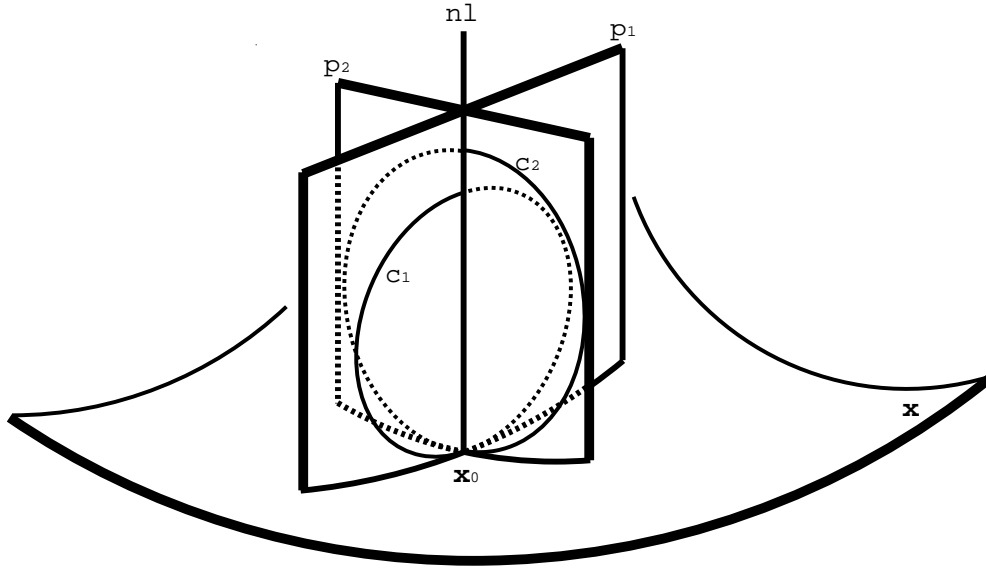


Figure 5.2: Configuration for the critical point problem for isophotes - non-umbilical point.

**Remark 6:** For  $\kappa_1 = 0$  ( $\kappa_2 = 0$  respectively) the solution circle  $c_1$  ( $c_2$  respectively) turns out in the set of all directions in  $p_1$  ( $p_2$  respectively). If we define isophotes not by an eye point but by an eye direction  $\mathbf{r}$ , we obtain a critical point in  $\mathbf{x}_0$  iff at least one of the following conditions is satisfied:

- $\mathbf{r}$  is parallel to  $\mathbf{n}_0$ ,
- $\kappa_1 = \kappa_2 = 0$  (flat point),
- $\kappa_1 = 0$ ,  $\kappa_2 \neq 0$  and  $\mathbf{r}$  is in  $p_1$ ,
- $\kappa_1 \neq 0$ ,  $\kappa_2 = 0$  and  $\mathbf{r}$  is in  $p_2$ .

**”Thickness”:**

Since we know the scalar field  $s$ , we can use (4.38) and (4.39) to compute the ”thickness” of the isophotes.

**Continuity:**

Given are two surfaces  $\mathbf{x}$  and  $\tilde{\mathbf{x}}$  which join along a common line  $l$ . A family of isophotes which have in no point of  $l$  a critical point is  $G^2$  across  $l$  if  $\mathbf{x}$  and  $\tilde{\mathbf{x}}$  are  $G^3$  along  $l$ .

Since the curvature of isophotes contains only the partial derivatives of  $\mathbf{x}$

and  $\tilde{\mathbf{x}}$  of the order  $\leq 3$ , this property is obvious (and even formulated for  $G^r$  surfaces and  $G^{r-1}$  isophotes in [23]). Sufficient conditions for  $G^3$  of surfaces – based on  $G^2$  of isophotes – are still unknown.

## 5.5 Reflection Lines

Reflection lines ([17],[16]) are a standard surface interrogation tool in car design.

Given is a surface  $\mathbf{x}$ , an eye point  $\mathbf{e}$ , a plane and a family of parallel straight lines in the plane. The plane should be called *light plane* and can be described in two forms: as a point  $\mathbf{p}_0$  and two orthonormalized vectors  $\mathbf{p}_1$  and  $\mathbf{p}_2$ , or simply as an (unnormalized) vector  $\mathbf{p}$ . In the first case the plane contains all points  $\mathbf{p}_0 + \lambda \cdot \mathbf{p}_1 + \mu \cdot \mathbf{p}_2$  ( $\lambda, \mu \in \mathbb{R}$ ), in the second case the plane is defined as containing the point  $\mathbf{e} + \mathbf{p}$  and being perpendicular to  $\mathbf{p}$ . Here we want to use both forms of describing the plane.

The surface  $\mathbf{x}$  is considered mirror-like. Reflection lines on the surface  $\mathbf{x}$  are the mirror image of the family of straight lines in the plane while looking from the eye point  $\mathbf{e}$  (see figure 5.3).

The definition of reflection lines depends on a particular *configuration*. This configuration contains the location of the eye point, the light plane and the direction of the lines in the light plane. We want to compute the curvature of the reflection lines for a given surface and a given configuration. The light plane is given by a point  $\mathbf{p}_0$  and two orthonormalized vectors  $\mathbf{p}_1$  and  $\mathbf{p}_2$ . The family of straight lines is given by two numbers  $cx$  and  $cy$  which satisfy  $cx^2 + cy^2 = 1$  and contains all lines in the light plane which are in the direction  $cx \cdot \mathbf{p}_1 + cy \cdot \mathbf{p}_2$ .

Let  $D$  be the domain of the surface  $\mathbf{x}$ . We define another surface  $\mathbf{y}$  over  $D$  in the following way:

For every point  $(u, v) \in D$ , we take  $\mathbf{x}(u, v)$ , compute the surface normal  $\mathbf{n}(u, v)$  in  $\mathbf{x}(u, v)$ , compute the reflected ray  $\mathbf{a}$  of  $\mathbf{x} - \mathbf{e}$  in the tangent plane of  $\mathbf{x}(u, v)$  and intersect this ray with the light plane given in the configuration. The intersection point of  $\mathbf{a}$  and the light plane is considered as  $\mathbf{y}(u, v)$  (see figure 5.4)

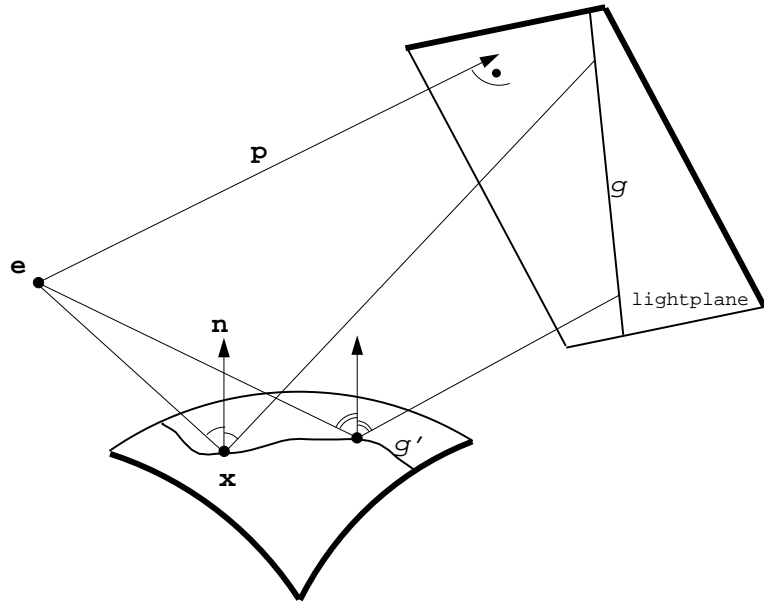


Figure 5.3: Reflection lines:  $g'$  is the mirror image of the straight line  $g$  on  $\mathbf{x}$

The surface  $\mathbf{y}$  lies completely in the light plane. Therefore, all partial derivative vectors of  $\mathbf{y}$  are in the light plane as well.

Now we want to develop the formula of  $\mathbf{y}$  and its partial derivatives. The reflected ray  $\mathbf{a}$  can be described in the form

$$\mathbf{a} = 2 \cdot ((\mathbf{e} - \mathbf{x}) \cdot \mathbf{n}) \cdot \mathbf{n} + \mathbf{x} - \mathbf{e}. \quad (5.103)$$

This yields for the partial derivatives of  $\mathbf{a}$ :

$$\begin{aligned} \mathbf{a}_u = & 2 \cdot ((\mathbf{e} - \mathbf{x}) \cdot \mathbf{n}_u) \cdot \mathbf{n} \\ & + 2 \cdot ((\mathbf{e} - \mathbf{x}) \cdot \mathbf{n}) \cdot \mathbf{n}_u \\ & + \mathbf{x}_u \end{aligned} \quad (5.104)$$

$$\begin{aligned} \mathbf{a}_v = & 2 \cdot ((\mathbf{e} - \mathbf{x}) \cdot \mathbf{n}_v) \cdot \mathbf{n} \\ & + 2 \cdot ((\mathbf{e} - \mathbf{x}) \cdot \mathbf{n}) \cdot \mathbf{n}_v \\ & + \mathbf{x}_v \end{aligned} \quad (5.105)$$



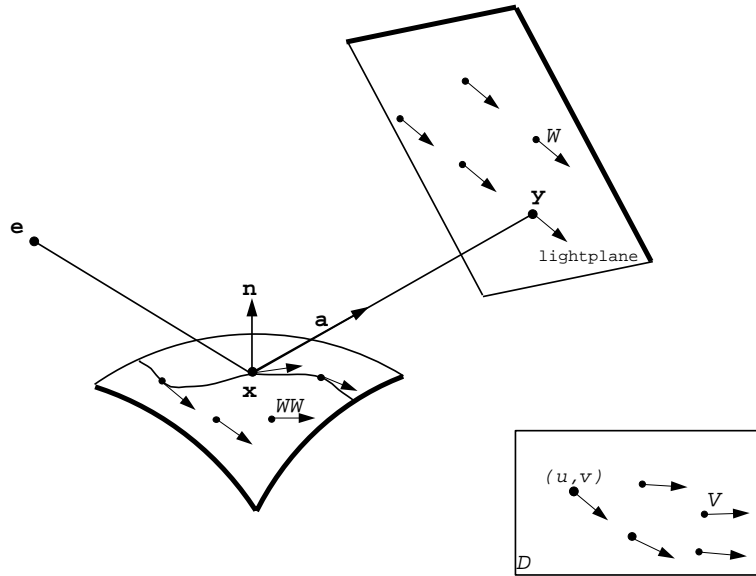


Figure 5.4: Reflection lines: definition of the surface  $y$  and the corresponding vector fields  $W$ ,  $V$  and  $WW$

$$\begin{aligned}
 \mathbf{a}_{uu} = & 2 \cdot ((\mathbf{e} - \mathbf{x}) \cdot \mathbf{n}_{uu}) \cdot \mathbf{n} \\
 & + 4 \cdot ((\mathbf{e} - \mathbf{x}) \cdot \mathbf{n}_u) \cdot \mathbf{n}_u \\
 & + 2 \cdot ((\mathbf{e} - \mathbf{x}) \cdot \mathbf{n}) \cdot \mathbf{n}_{uu} \\
 & + 2 \cdot (\mathbf{x}_{uu} \cdot \mathbf{n}) \cdot \mathbf{n} \\
 & + \mathbf{x}_{uu}
 \end{aligned} \tag{5.106}$$

$$\begin{aligned}
 \mathbf{a}_{uv} = & 2 \cdot ((\mathbf{e} - \mathbf{x}) \cdot \mathbf{n}_{uv}) \cdot \mathbf{n} \\
 & + 2 \cdot ((\mathbf{e} - \mathbf{x}) \cdot \mathbf{n}_v) \cdot \mathbf{n}_u \\
 & + 2 \cdot ((\mathbf{e} - \mathbf{x}) \cdot \mathbf{n}_u) \cdot \mathbf{n}_v \\
 & + 2 \cdot ((\mathbf{e} - \mathbf{x}) \cdot \mathbf{n}) \cdot \mathbf{n}_{uv} \\
 & + 2 \cdot (\mathbf{x}_{uv} \cdot \mathbf{n}) \cdot \mathbf{n} \\
 & + \mathbf{x}_{uv}
 \end{aligned} \tag{5.107}$$

$$\begin{aligned}
\mathbf{a}_{vv} = & 2 \cdot ((\mathbf{e} - \mathbf{x}) \cdot \mathbf{n}_{vv}) \cdot \mathbf{n} \\
& + 4 \cdot ((\mathbf{e} - \mathbf{x}) \cdot \mathbf{n}_v) \cdot \mathbf{n}_v \\
& + 2 \cdot ((\mathbf{e} - \mathbf{x}) \cdot \mathbf{n}) \cdot \mathbf{n}_{vv} \\
& + 2 \cdot (\mathbf{x}_{vv} \cdot \mathbf{n}) \cdot \mathbf{n} \\
& + \mathbf{x}_{vv}.
\end{aligned} \tag{5.108}$$

The light plane is given by a point  $\mathbf{p}_0$  and two vectors  $\mathbf{p}_1$  and  $\mathbf{p}_2$ . To intersect the reflected ray with the light plane, we have to solve the system of equations  $\mathbf{x} + \alpha \cdot \mathbf{a} = \mathbf{p}_0 + \beta \cdot \mathbf{p}_1 + \gamma \cdot \mathbf{p}_2$  for  $\alpha$ ,  $\beta$  and  $\gamma$ . (We assume that the reflected ray  $\mathbf{a}$  is not parallel to the plane, i.e. there is a unique solution of the system of equations.) We obtain for  $\alpha$ :

$$\alpha = \frac{\det[\mathbf{p}_0 - \mathbf{x}, \mathbf{p}_1, \mathbf{p}_2]}{\det[\mathbf{a}, \mathbf{p}_1, \mathbf{p}_2]}. \tag{5.109}$$

Using the abbreviation  $\mathbf{p}_3 := \mathbf{p}_1 \times \mathbf{p}_2$ , we can write  $\alpha$  in the following form:

$$\alpha = \frac{(\mathbf{p}_0 - \mathbf{x}) \cdot \mathbf{p}_3}{\mathbf{a} \cdot \mathbf{p}_3}. \tag{5.110}$$

This yields for the partial derivatives of  $\alpha$ :

$$\alpha_u = -\frac{\mathbf{x}_u \cdot \mathbf{p}_3}{\mathbf{a} \cdot \mathbf{p}_3} - \frac{(\mathbf{a}_u \cdot \mathbf{p}_3) \cdot ((\mathbf{p}_0 - \mathbf{x}) \cdot \mathbf{p}_3)}{(\mathbf{a} \cdot \mathbf{p}_3)^2} \tag{5.111}$$

$$\alpha_v = -\frac{\mathbf{x}_v \cdot \mathbf{p}_3}{\mathbf{a} \cdot \mathbf{p}_3} - \frac{(\mathbf{a}_v \cdot \mathbf{p}_3) \cdot ((\mathbf{p}_0 - \mathbf{x}) \cdot \mathbf{p}_3)}{(\mathbf{a} \cdot \mathbf{p}_3)^2} \tag{5.112}$$

$$\begin{aligned}
\alpha_{uu} = & -\frac{\mathbf{x}_{uu} \cdot \mathbf{p}_3}{\mathbf{a} \cdot \mathbf{p}_3} + 2 \cdot \frac{((\mathbf{p}_0 - \mathbf{x}) \cdot \mathbf{p}_3) \cdot (\mathbf{a}_u \cdot \mathbf{p}_3)^2}{(\mathbf{a} \cdot \mathbf{p}_3)^3} \\
& + \frac{2 \cdot (\mathbf{x}_u \cdot \mathbf{p}_3) \cdot (\mathbf{a}_u \cdot \mathbf{p}_3) - ((\mathbf{p}_0 - \mathbf{x}) \cdot \mathbf{p}_3) \cdot (\mathbf{a}_{uu} \cdot \mathbf{p}_3)}{(\mathbf{a} \cdot \mathbf{p}_3)^2}
\end{aligned} \tag{5.113}$$

$$\begin{aligned}
\alpha_{uv} = & -\frac{\mathbf{x}_{uv} \cdot \mathbf{p}_3}{\mathbf{a} \cdot \mathbf{p}_3} - \frac{((\mathbf{p}_0 - \mathbf{x}) \cdot \mathbf{p}_3) \cdot (\mathbf{a}_{uv} \cdot \mathbf{p}_3)}{(\mathbf{a} \cdot \mathbf{p}_3)^2} \\
& + \frac{(\mathbf{x}_u \cdot \mathbf{p}_3) \cdot (\mathbf{a}_v \cdot \mathbf{p}_3) + (\mathbf{x}_v \cdot \mathbf{p}_3) \cdot (\mathbf{a}_u \cdot \mathbf{p}_3)}{(\mathbf{a} \cdot \mathbf{p}_3)^2} \\
& + 2 \cdot \frac{((\mathbf{p}_0 - \mathbf{x}) \cdot \mathbf{p}_3) \cdot (\mathbf{a}_u \cdot \mathbf{p}_3) \cdot (\mathbf{a}_v \cdot \mathbf{p}_3)}{(\mathbf{a} \cdot \mathbf{p}_3)^3}
\end{aligned} \tag{5.114}$$

$$\alpha_{vv} = - \frac{\mathbf{x}_{vv} \cdot \mathbf{p}_3}{\mathbf{a} \cdot \mathbf{p}_3} + 2 \cdot \frac{((\mathbf{p}_0 - \mathbf{x}) \cdot \mathbf{p}_3) \cdot (\mathbf{a}_v \cdot \mathbf{p}_3)^2}{(\mathbf{a} \cdot \mathbf{p}_3)^3} + \frac{2 \cdot (\mathbf{x}_v \cdot \mathbf{p}_3) \cdot (\mathbf{a}_v \cdot \mathbf{p}_3) - ((\mathbf{p}_0 - \mathbf{x}) \cdot \mathbf{p}_3) \cdot (\mathbf{a}_{vv} \cdot \mathbf{p}_3)}{(\mathbf{a} \cdot \mathbf{p}_3)^2}. \quad (5.115)$$

Now we have for  $\mathbf{y}$ :

$$\mathbf{y} = \mathbf{x} + \alpha \cdot \mathbf{a} \quad (5.116)$$

which yields for the partial derivatives:

$$\mathbf{y}_u = \mathbf{x}_u + \alpha_u \cdot \mathbf{a} + \alpha \cdot \mathbf{a}_u \quad (5.117)$$

$$\mathbf{y}_v = \mathbf{x}_v + \alpha_v \cdot \mathbf{a} + \alpha \cdot \mathbf{a}_v \quad (5.118)$$

$$\mathbf{y}_{uu} = \mathbf{x}_{uu} + \alpha_{uu} \cdot \mathbf{a} + 2 \cdot \alpha_u \cdot \mathbf{a}_u + \alpha \cdot \mathbf{a}_{uu} \quad (5.119)$$

$$\mathbf{y}_{uv} = \mathbf{x}_{uv} + \alpha_{uv} \cdot \mathbf{a} + \alpha_u \cdot \mathbf{a}_v + \alpha_v \cdot \mathbf{a}_u + \alpha \cdot \mathbf{a}_{uv} \quad (5.120)$$

$$\mathbf{y}_{vv} = \mathbf{x}_{vv} + \alpha_{vv} \cdot \mathbf{a} + 2 \cdot \alpha_v \cdot \mathbf{a}_v + \alpha \cdot \mathbf{a}_{vv}. \quad (5.121)$$

Now we consider the vector field  $W$  over  $\mathbf{y}$  which is defined by

$$W = cx \cdot \mathbf{p}_1 + cy \cdot \mathbf{p}_2 \quad (5.122)$$

This constant vector field produces the family of straight lines as tangent curves. Using the results from section 4.2, we can easily compute a vector field  $V$  in the domain which is corresponding to  $W$ :

$$\begin{aligned} V &= \begin{pmatrix} vx \\ vy \end{pmatrix} = \begin{pmatrix} \det[\mathbf{p}_1 \times \mathbf{p}_2, cx \cdot \mathbf{p}_1 + cy \cdot \mathbf{p}_2, \mathbf{y}_v] \\ -\det[\mathbf{p}_1 \times \mathbf{p}_2, cx \cdot \mathbf{p}_1 + cy \cdot \mathbf{p}_2, \mathbf{y}_u] \end{pmatrix} \\ &= cx \cdot \begin{pmatrix} \det[\mathbf{p}_1 \times \mathbf{p}_2, \mathbf{p}_1, \mathbf{y}_v] \\ -\det[\mathbf{p}_1 \times \mathbf{p}_2, \mathbf{p}_1, \mathbf{y}_u] \end{pmatrix} \\ &\quad + cy \cdot \begin{pmatrix} \det[\mathbf{p}_1 \times \mathbf{p}_2, \mathbf{p}_2, \mathbf{y}_v] \\ -\det[\mathbf{p}_1 \times \mathbf{p}_2, \mathbf{p}_2, \mathbf{y}_u] \end{pmatrix}. \end{aligned} \quad (5.123)$$

Using the assumption that  $\mathbf{p}_1$  and  $\mathbf{p}_2$  are orthonormalized, we can write  $V$  in the form

$$V = cx \cdot \begin{pmatrix} \mathbf{p}_2 \cdot \mathbf{y}_v \\ -\mathbf{p}_2 \cdot \mathbf{y}_u \end{pmatrix} + cy \cdot \begin{pmatrix} -\mathbf{p}_1 \cdot \mathbf{y}_v \\ \mathbf{p}_1 \cdot \mathbf{y}_u \end{pmatrix}. \quad (5.124)$$

This gives for the partial derivatives:

$$V_u = cx \cdot \begin{pmatrix} \mathbf{p}_2 \cdot \mathbf{y}_{uv} \\ -\mathbf{p}_2 \cdot \mathbf{y}_{uu} \end{pmatrix} + cy \cdot \begin{pmatrix} -\mathbf{p}_1 \cdot \mathbf{y}_{uv} \\ \mathbf{p}_1 \cdot \mathbf{y}_{uu} \end{pmatrix} \quad (5.125)$$

$$V_v = cx \cdot \begin{pmatrix} \mathbf{p}_2 \cdot \mathbf{y}_{vv} \\ -\mathbf{p}_2 \cdot \mathbf{y}_{uv} \end{pmatrix} + cy \cdot \begin{pmatrix} -\mathbf{p}_1 \cdot \mathbf{y}_{vv} \\ \mathbf{p}_1 \cdot \mathbf{y}_{uv} \end{pmatrix}. \quad (5.126)$$

Now we can compute a vector field  $WW$  over  $\mathbf{x}$  which is corresponding to  $V$  in the domain. The tangent curves of  $WW$  are identically to the reflection lines defined above. Knowing  $V$ ,  $V_u$  and  $V_v$ , we can easily compute their curvature.

### Critical points:

The appearance of critical points on the surface  $\mathbf{x}$  depends on the particular configuration. We want to consider two kinds of critical points: first and second order critical points.

Given is the surface  $\mathbf{x}$ , an eye point  $\mathbf{e} = (ex, ey, ez)^T$  and a light plane (now defined by a vector  $\mathbf{p} = (px, py, pz)^T$  with  $\mathbf{p} \neq \mathbf{0}$ ). We define:

$\mathbf{x}$  has a *first order critical point* in  $\mathbf{x}_0$  iff for every family of parallel lines in the light plane the reflection line vector field has a critical point in  $\mathbf{x}_0$ .  $\mathbf{x}$  has a *second order critical point* in  $\mathbf{x}_0$  iff there is a family of parallel lines in the light plane which produces a critical point in the reflection line vector field in  $\mathbf{x}_0$ .

The condition for a critical point is  $V = (0, 0)^T$ . This, equation (5.124), the assumption  $cx^2 + cy^2 = 1$ , and the assumption of orthonormalization of  $\mathbf{p}_1$  and  $\mathbf{p}_2$  yields that we only have to consider  $\mathbf{y}_u$  and  $\mathbf{y}_v$ . We have a second order critical point iff  $\mathbf{y}_u$  and  $\mathbf{y}_v$  are linearly dependent. (In fact, if we have  $\mathbf{y}_v = \lambda \cdot \mathbf{y}_u$  and  $\mathbf{y}_u \neq \mathbf{0}$ , we set  $cx := \frac{\mathbf{p}_1 \cdot \mathbf{y}_u}{\sqrt{(\mathbf{p}_1 \cdot \mathbf{y}_u)^2 + (\mathbf{p}_2 \cdot \mathbf{y}_u)^2}}$  and  $cy := \frac{\mathbf{p}_2 \cdot \mathbf{y}_u}{\sqrt{(\mathbf{p}_1 \cdot \mathbf{y}_u)^2 + (\mathbf{p}_2 \cdot \mathbf{y}_u)^2}}$ . This obviously yields  $V = (0, 0)^T$ .)

Therefore, the necessary and sufficient condition for a second order critical point is

$$\mathbf{y}_u \times \mathbf{y}_v = \mathbf{0}. \quad (5.127)$$

The necessary and sufficient condition for a first order critical point is

$$\mathbf{y}_u = \mathbf{0} \text{ and } \mathbf{y}_v = \mathbf{0}. \quad (5.128)$$

We assume that the reflected ray in  $\mathbf{x}$  is not parallel to the plane, i.e.

$$\mathbf{a} \cdot \mathbf{p} \neq 0. \quad (5.129)$$

Furthermore, we use the following abbreviations:

$$h_1 = \mathbf{p}^2 + 2 \cdot ((\mathbf{e} - \mathbf{x}) \cdot \mathbf{n}) \cdot (\mathbf{p} \cdot \mathbf{n}) \quad (5.130)$$

$$h_2 = (\mathbf{e} - \mathbf{x}) \cdot \mathbf{p} + \mathbf{p}^2. \quad (5.131)$$

We start with the treatment of first order critical points.

We assume that the surface is locally arc length parametrized by the lines of curvature. Furthermore, we assume  $\mathbf{x} = \mathbf{0}$  at the considered point. This yields (5.90)-(5.97) for  $\mathbf{x} = \mathbf{x}_0$  and

$$\mathbf{p} \cdot \mathbf{n} = pz \quad (5.132)$$

$$\mathbf{a} = (-ex, -ey, ez)^T. \quad (5.133)$$

Then we can write  $\mathbf{y}_u$  and  $\mathbf{y}_v$  in the form:

$$\mathbf{y}_u = \frac{1}{(\mathbf{a} \cdot \mathbf{p})^2} \cdot (h_1 \cdot G_1 + 2 \cdot h_2 \cdot \kappa_1 \cdot G_2) \quad (5.134)$$

$$\mathbf{y}_v = \frac{1}{(\mathbf{a} \cdot \mathbf{p})^2} \cdot (h_1 \cdot G_3 + 2 \cdot h_2 \cdot \kappa_2 \cdot G_4) \quad (5.135)$$

where  $\kappa_1$  and  $\kappa_2$  are the principal curvatures in the principal directions  $\mathbf{x}_u$  and  $\mathbf{x}_v$  and

$$G_1 = \begin{pmatrix} ez \cdot pz - ey \cdot py \\ px \cdot ey \\ -ez \cdot px \end{pmatrix} \quad (5.136)$$

$$G_2 = \begin{pmatrix} ey \cdot ez \cdot py - ex^2 \cdot pz - ez^2 \cdot pz \\ -(ez \cdot px + ex \cdot pz) \cdot ey \\ px \cdot ex^2 + px \cdot ez^2 + ex \cdot ey \cdot py \end{pmatrix} \quad (5.137)$$

$$G_3 = \begin{pmatrix} ex \cdot py \\ ez \cdot pz - ex \cdot px \\ -ez \cdot py \end{pmatrix} \quad (5.138)$$

$$G_4 = \begin{pmatrix} -(ey \cdot pz + ez \cdot py) \cdot ex \\ ez \cdot ex \cdot px - ey^2 \cdot pz - ez^2 \cdot pz \\ ey \cdot ex \cdot px + ey^2 \cdot py + py \cdot ez^2 \end{pmatrix}. \quad (5.139)$$

From (5.133) and (5.136) - (5.139) we obtain

$$-G_1 \times G_2 = G_3 \times G_4 = (ex \cdot ey \cdot (\mathbf{a} \cdot \mathbf{p})) \cdot \mathbf{p}. \quad (5.140)$$

We make a case distinction:

**case 1:**  $ex \neq 0$  and  $ey \neq 0$ .

We know from (5.140) and (5.129) that  $G_1$  and  $G_2$  are linearly independent. Therefore, to get  $\mathbf{y}_u = \mathbf{0}$  we have to achieve

$$h_1 = 0 \quad (5.141)$$

$$\text{and } h_2 \cdot \kappa_1 = 0. \quad (5.142)$$

Since  $h_1 - h_2 = \mathbf{a} \cdot \mathbf{p}$  and (5.129), we get for (5.142):

$$\kappa_1 = 0. \quad (5.143)$$

In a similar way we obtain the conditions for  $\mathbf{y}_v = 0$ : (5.141) and

$$\kappa_2 = 0. \quad (5.144)$$

We thus have a first order critical point at a flat point satisfying (5.141).

**case 2:**  $ex = 0$  and  $ey \neq 0$ .

In this case we have

$$\mathbf{y}_u = \frac{h_1 - 2 \cdot \kappa_1 \cdot ez \cdot h_2}{(\mathbf{a} \cdot \mathbf{p})^2} \cdot G_1 \quad (5.145)$$

$$\mathbf{y}_v = \frac{ez \cdot h_1 - 2 \cdot \kappa_2 \cdot (\mathbf{e} - \mathbf{x})^2 \cdot h_2}{(\mathbf{a} \cdot \mathbf{p})^2} \cdot \begin{pmatrix} 0 \\ pz \\ -py \end{pmatrix}. \quad (5.146)$$

Since we know (5.129) and  $\|G_1\|^2 = (\mathbf{a} \cdot \mathbf{p})^2 + px^2 \cdot (\mathbf{e} - \mathbf{x})^2 \neq 0$ , we get the conditions for  $\mathbf{y}_u = 0$  and  $\mathbf{y}_v = 0$ :

$$h_1 = 2 \cdot \kappa_1 \cdot ez \cdot h_2 \quad (5.147)$$

$$ez \cdot h_1 = 2 \cdot \kappa_2 \cdot (\mathbf{e} - \mathbf{x})^2 \cdot h_2 \quad (5.148)$$

**case 3:**  $ex \neq 0$  and  $ey = 0$ .

similar to case 2, we obtain the conditions

$$h_1 = 2 \cdot \kappa_2 \cdot ez \cdot h_2 \quad (5.149)$$

$$ez \cdot h_1 = 2 \cdot \kappa_1 \cdot (\mathbf{e} - \mathbf{x})^2 \cdot h_2. \quad (5.150)$$

**case 4:**  $ex = 0$  and  $ey = 0$ .

can be considered as a special case of case 2 or case 3. We obtain the conditions (5.147) and (5.149).

The result of our case distinction is the following: For every point on the surface we can find an appropriate configuration so that this point is a first order critical point. (We even can choose  $\mathbf{p}$  arbitrary. If the surface point is a flat point, we can use case 1 for computing a suitable eye point. If the surface point is an umbilical (but not flat) point, we can use case 4. For all other surface points we can use case 2 or 3 to find a suitable eye point.)

Now we want to treat second order critical points. Using the assumption of locally arc length parametrization by the lines of curvature and  $\mathbf{x} = \mathbf{0}$  at the considered point, we can write  $\mathbf{y}_u \times \mathbf{y}_v$  in the following form:

$$\mathbf{y}_u \times \mathbf{y}_v = \frac{f}{(\mathbf{a} \cdot \mathbf{p})^3} \cdot \mathbf{p} \quad (5.151)$$

where

$$\begin{aligned} f = & \quad ez \cdot h_1^2 \\ & - 2 \cdot (\kappa_1 \cdot ex^2 + \kappa_2 \cdot ey^2) \cdot h_1 \cdot h_2 \\ & - 2 \cdot (\kappa_1 + \kappa_2) \cdot ez^2 \cdot h_1 \cdot h_2 \\ & + 4 \cdot ez \cdot \kappa_1 \cdot \kappa_2 \cdot (\mathbf{e} - \mathbf{x})^2 \cdot h_2^2. \end{aligned} \quad (5.152)$$

Let  $K := \kappa_1 \cdot \kappa_2$  be the Gaussian curvature and  $H := (\kappa_1 + \kappa_2)/2$  be the mean curvature of  $\mathbf{x}$ . Furthermore, let  $\kappa$  be the normal curvature of  $\mathbf{x}$  in the direction given by the projection of  $\mathbf{e} - \mathbf{x}$  into the tangent plane of  $\mathbf{x}$ . Then Euler's theorem yields  $\kappa = \frac{\kappa_1 \cdot ex^2 + \kappa_2 \cdot ey^2}{\sqrt{ex^2 + ey^2}}$  and we can write  $f$  in the form

$$\begin{aligned} f = & \quad ez \cdot h_1^2 \\ & - 2 \cdot (\kappa \cdot \sqrt{(\mathbf{e} - \mathbf{x})^2 - ez^2} + 2 \cdot H \cdot ez^2) \cdot h_1 \cdot h_2 \\ & + 4 \cdot ez \cdot K \cdot (\mathbf{e} - \mathbf{x})^2 \cdot h_2^2. \end{aligned} \quad (5.153)$$

The necessary and sufficient condition for a second order critical point is

$$f = 0. \quad (5.154)$$

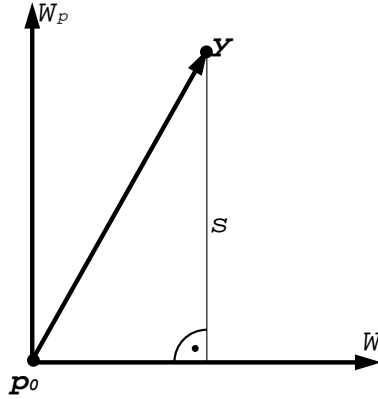


Figure 5.5: Reflection lines as a scalar field. Shown is the light plane.  $s$  is the distance of  $\mathbf{y}$  to the line  $\mathbf{p}_0 + \lambda \cdot \mathbf{W}$ . Since we know  $\|\mathbf{W}\| = \|\mathbf{W}_p\| = 1$  and  $\mathbf{W} \cdot \mathbf{W}_p = 0$ , we obtain  $s = \mathbf{W}_p \cdot (\mathbf{y} - \mathbf{p}_0)$

**”Thickness:”**

The ”thickness” of reflection lines has a nice practical meaning: analyzing a surface using reflection lines the designer moves his/her eye and observes how fast the reflection lines ”move” on the surface. The ”thickness” is a measure of how fast the reflection lines are moving.

We can consider reflection lines as obtained by a scalar field. If we define  $\mathbf{W}_p$  as the perpendicular vector of  $\mathbf{W}$  (using equation (5.122)) in the light plane, we obtain

$$\mathbf{W}_p = -cy \cdot \mathbf{p}_1 + cx \cdot \mathbf{p}_2. \tag{5.155}$$

Then we can write the scalar field in the form

$$s = \mathbf{W}_p \cdot (\mathbf{y} - \mathbf{p}_0) \tag{5.156}$$

$$s_u = \mathbf{W}_p \cdot \mathbf{y}_u \tag{5.157}$$

$$s_v = \mathbf{W}_p \cdot \mathbf{y}_v \tag{5.158}$$

Now we can use (4.38) and (4.39) to compute the ”thickness” of reflection lines. See figure 5.5 for an illustration.

**Continuity:**

Given are two surfaces  $\mathbf{x}$  and  $\tilde{\mathbf{x}}$  which join along a common line  $l$ . A family



of reflection lines which have in no point of  $l$  a critical point is  $G^2$  across  $l$  if  $\mathbf{x}$  and  $\tilde{\mathbf{x}}$  are  $G^3$  along  $l$ .  
Since the curvature of reflection lines contains only the partial derivatives of  $\mathbf{x}$  and  $\tilde{\mathbf{x}}$  of the order  $\leq 3$ , this property is obvious.  
Sufficient conditions for  $G^3$  of surfaces – based on  $G^2$  of reflection lines – are still unknown.

## Chapter 6

# Tangent Curves and Surface Interrogation

A surface designed by a CAD-system may look perfect in the wire frame (and even in the shaded) representation. Nevertheless the surface can have imperfections, an undesired behavior of characteristic properties, or areas which simply do not look "nice". It is the task of *surface interrogation algorithms* to point out those imperfections on the surface.

Surface interrogation algorithms focus on the following two points:

- a)** point out geometric properties of the surface which usually can be described in mathematical terms. These properties can be:
- geometric continuity on the patch borders,
  - special points on the surface (flat points, umbilical points...),
  - convexity properties (change of the sign of the curvatures),
  - strong and frequent variations of the curvatures.
- b)** Give a global impression about smoothness and fairness of the surface.

A variety of surface interrogation algorithms have been developed which emphasize different aspects of the points a) and/or b). A survey on surface interrogation algorithms is given in [8] and [9].

The tangent curves on surfaces discussed in chapter 5 are wellknown as standard surface interrogation tools. In this chapter we want to discuss the additional usage of the curvature plots and (if possible) the "thickness" plots of those curves as surface interrogation tools. This gives more information

about the geometric properties of the surface. (In fact, the usual application of those tangent curves gives us only first and second order information (i.e., properties which contain only the first and second partial derivatives of the surface). Curvature visualization gives additional third order information.)

**The test surface:**

Figure 6.1 shows the ray traced image of the test surface – a shoe shaped (non-rational) bicubic B-spline surface. This surface consists of  $29 \times 10$  patches and is  $C^2$  at the patch borders. We hardly can see surface imperfections in the picture. Also a wire frame representation of this surface would look perfect.

**Tangent curve "thickness" for surface interrogation:**

We compute the "thickness" of the tangent curves for an appropriate number of surface points and color code these values as shown in figure 3.1.<sup>1</sup> Using the ray tracing approach it is easy to pick out an appropriate number of surface points: we have one surface point for every pixel point. For other rendering techniques we have to pick out sample points on the surface and apply an interpolation between them.

**Tangent curve curvature for surface interrogation:**

We color code the curvature of the tangent curves as shown in figure 3.1. Concerning the number of appropriate sample points, the same statement as for the tangent curve "thickness" applies. Instead of the tangent curve curvature we also can take the geodesic curvature of the tangent curves. This makes it possible to consider a sign in the curvature visualization.

Now we consider the particular tangent curves:

**Contour lines:**

We consider figure 6.2. The upper left picture shows the usual contour line visualization described in section 4.5. The upper right picture is the visualization of the contour line "thickness". The lower left picture shows the curvature plot of the contour lines, the lower right picture shows their geodesic curvature. All of the four pictures treat the same family of contour

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<sup>1</sup>For the following pictures the negative values of the "thickness" were actually taken. This way the "thickness" plots appear in a green color which looked better than in red.

lines.

The geodesic curvature plot shows frequent changes of the curvature sign (i.e., changes of red and green areas) in the right-hand part of the surface. That means, the contour lines in this area have inflection points. This is hardly detectable from the upper left picture.

The curvature visualization looks smooth, we can't see the patch borders. This is an indicator for  $G^2$ -continuity of the surface.

Since the location of the critical points of the contour lines depends on the particular configuration (i.e., on the direction vector  $\mathbf{r}$ ), the highlights in the "thickness" and curvature plots do not directly provide information about the surface.

### **Lines of curvature**

The upper two pictures of figure 6.3 show the geodesic curvature of the two families of lines of curvature. The lower two pictures are magnifications of the upper ones. The discontinuities at the patch borders show that the surface is not  $G^3$ . The umbilical points of the surface can be detected as highlights in the curvature plots.

### **Asymptotic lines**

The upper two pictures of figure 6.4 show the geodesic curvature of the two families of asymptotic lines, the lower two pictures are the magnifications. Since the asymptotic lines exist only for surface areas with non-positive Gaussian curvature, the areas with positive Gaussian curvature are set to the background color. The discontinuities on the patch borders show that the surface is not  $G^3$ .

### **Isophotes**

The upper left picture of figure 6.5 shows the visualization of a family of isophotes on the surface, the upper right picture shows their "thickness", the lower pictures show curvature and geodesic curvature of the isophotes. Figure 6.6 shows a magnification of the isophote visualization and the visualization of their geodesic curvature.

The "thickness" visualization clearly shows "wrinkles" (i.e., strip-shaped areas where bright and dark colors change rapidly). This shows a surface

imperfection in this area. The discontinuities on the patch borders show that the surface is not  $G^3$ .

We can use the critical points (i.e., the highlights in the curvature and "thickness" plots) to detect flat points on the surface: considering an eye direction (and not an eye point) and moving this eye direction around, a critical point on the surface might move around as well, or it might remain on the same position. A stationary highlight while moving the eye direction is an indicator for a flat point.

### **Reflection lines**

The upper left picture of figure 6.7 shows the visualization of a family of reflection lines on the surface, the upper right picture shows their "thickness", the lower pictures show their curvature and geodesic curvature. Figure 6.8 shows the magnification of the reflection line visualization and the geodesic curvature.

In the upper left picture of figure 6.7 we can see aliasing effects near the edges of the surface. These effects do not appear in the "thickness" and curvature plots because those plots contain only continuous color changes (except for the patch borders).

As in the case of isophotes, the "thickness" visualization of reflection lines shows "wrinkles" – approximately in the same areas as in the isophote "thickness" visualization. We also see the curvature discontinuities on the patch borders in the curvature plots.

The location of the critical points depends on the particular configuration, their appearance on the surface does not contain direct information about the surface geometry.

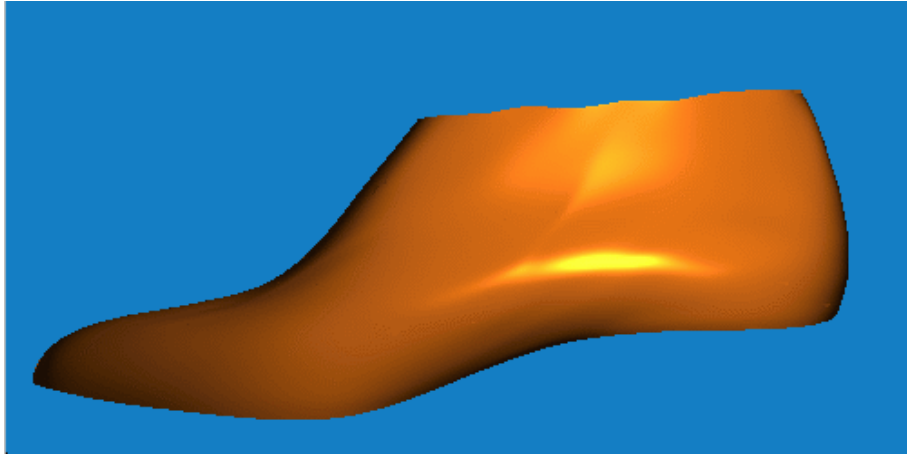


Figure 6.1: Ray traced test surface

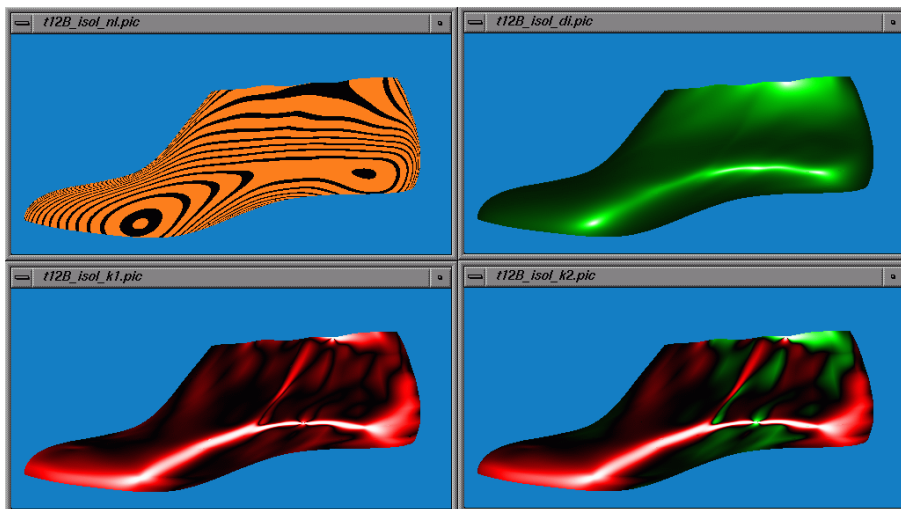


Figure 6.2: Contour lines, their "thickness" and curvature

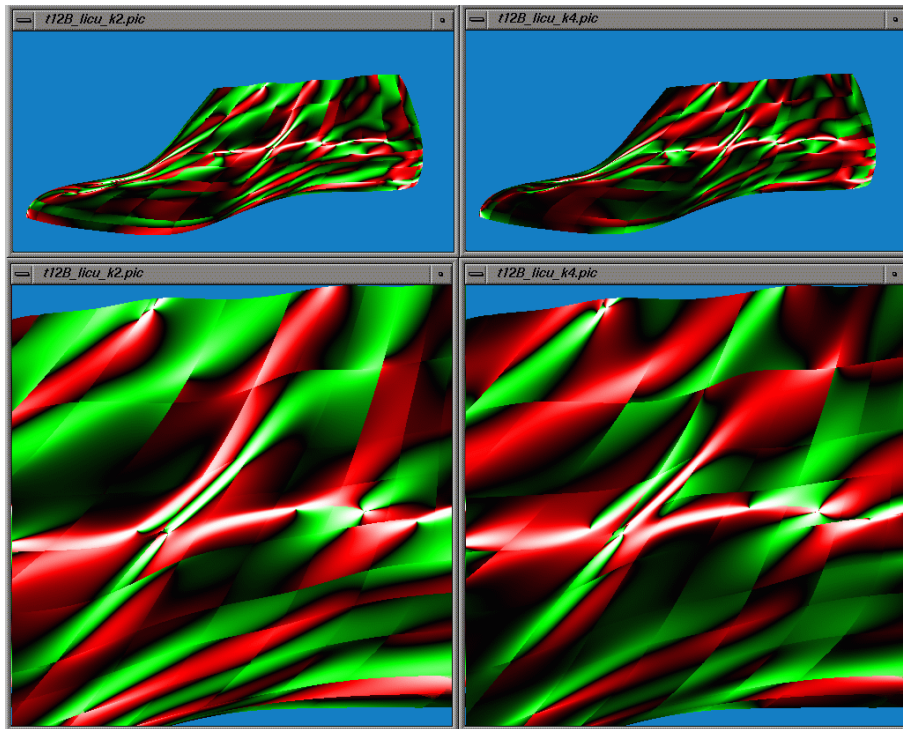


Figure 6.3: Geodesic curvature of the lines of curvature



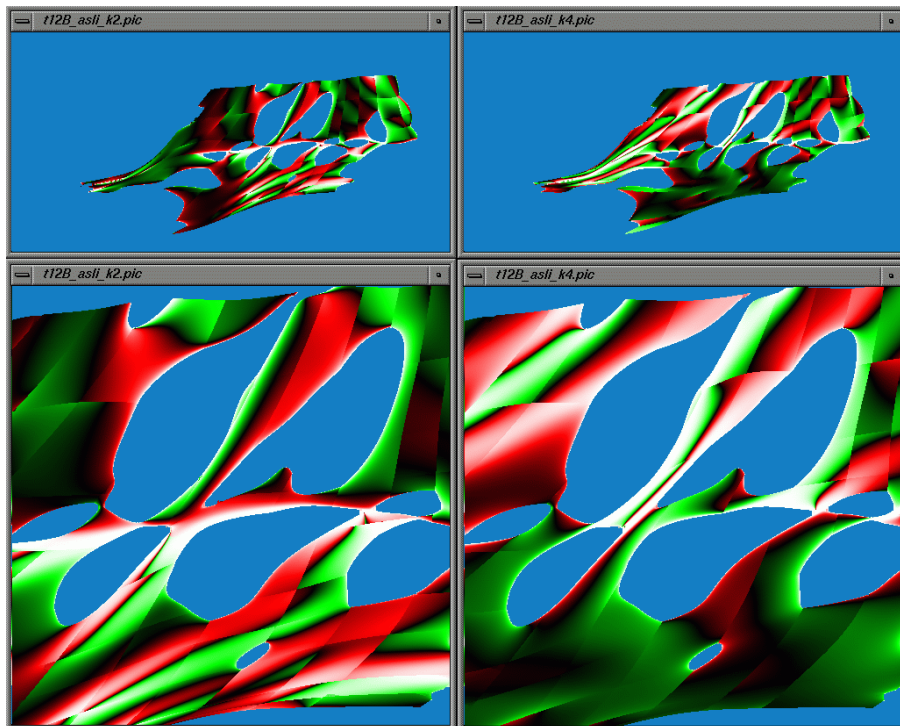


Figure 6.4: Geodesic curvature of the asymptotic lines

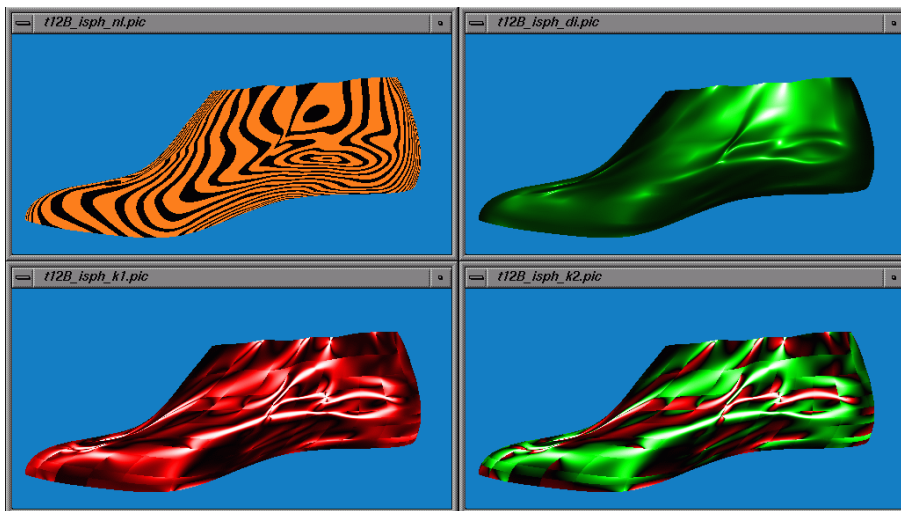


Figure 6.5: Isophotes, their "thickness" and curvature

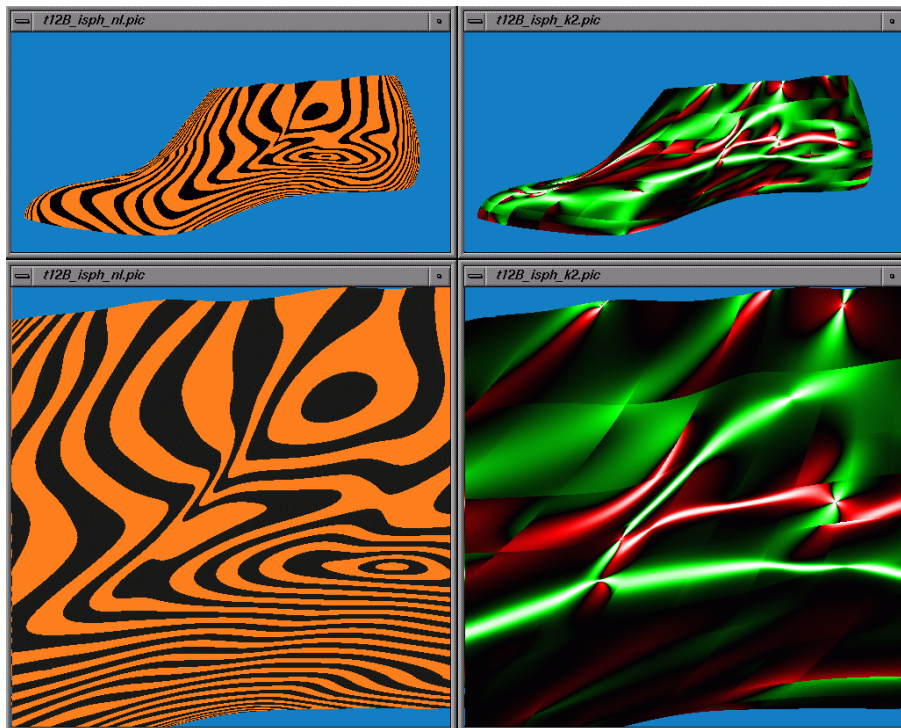


Figure 6.6: Isophotes and their geodesic curvature (magnification)

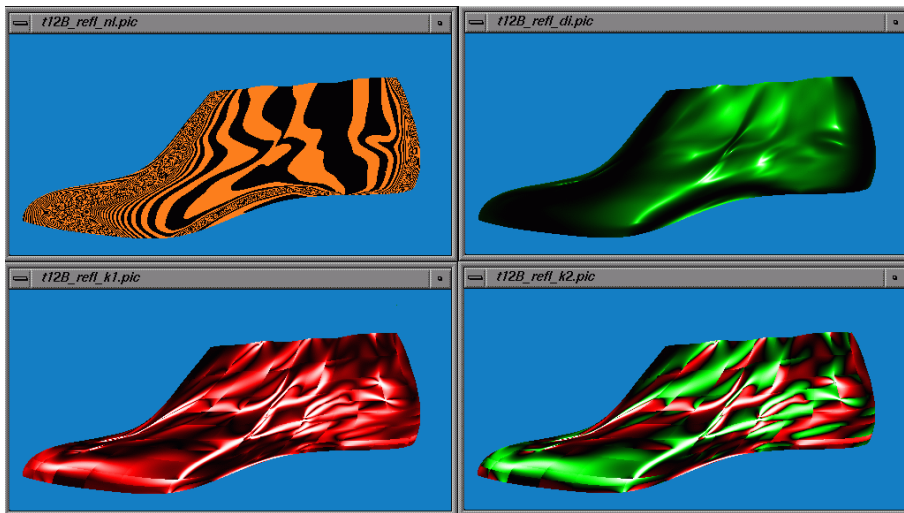


Figure 6.7: Reflection lines, their "thickness" and curvature

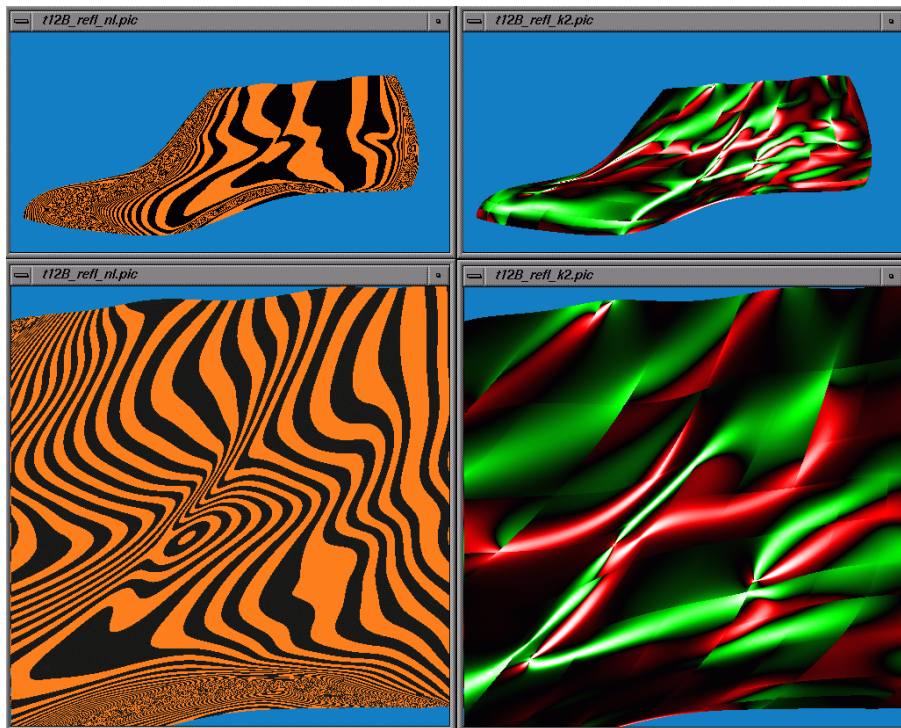


Figure 6.8: Reflection lines and their geodesic curvature (magnification)

# Chapter 7

## Open Questions

This work leaves the following open questions and thus material for future research:

- In theorem 2 we have shown that the curvature and the perpendicular curvature together contain all information about the normalized original vector field. Therefore, it makes sense to ask for an appropriate combination of both scalar fields from which we can recognize the behavior (i.e. for instance, the topology) of the vector field immediately. At least for the special case of linear vector fields this question is worth dealing with.
- The theory of vector field curvature should be extended to the 3D case. Some basic approaches are already given in section 2.8. Also, the visualization of the curvature of 3D vector fields requires further research.
- We should ask for sufficient conditions for  $G^3$  of surfaces – based on  $G^2$  of isophotes or reflection lines. In general, we should ask for sufficient conditions for  $G^r$  of surfaces based on  $G^{r-1}$  of lines of curvature, asymptotic lines, isophotes or reflection lines.
- The fact that we can compute the curvature of lines of curvature, asymptotic lines, isophotes and reflection lines on a surface exactly gives reason to try to use it as the base of surface fairing algorithms.

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## Hypotheses

1) *Tangent curves* (stream lines, characteristic lines) are a powerful tool for describing, analyzing and visualizing vector fields.

2) Although tangent curves are in general not describable as parametric curves, their curvatures can be computed for every point of the vector field (except for critical points). Thus, the *curvature of a vector field* can be defined as a scalar field which contains the curvature of the tangent curve for every point of the vector field. Using the concepts of normalized vector fields and vector field divergence, the curvature of vector fields can be expressed in a simple form.

3) The following *properties* are shown for the curvature of vector fields:

The rotated vector field of a 2D vector field  $V$  is obtained by rotating all vectors of  $V$  by a fixed angle. The curvature of a 2D vector field  $V$  and the curvature of its perpendicular (i.e. rotated by the angle  $\pi/2$ ) vector field give the curvatures of all rotated vector fields of  $V$ . Furthermore, the curvature and the perpendicular curvature of  $V$  define all vectors of  $V$  uniquely. In the neighborhood of a non-degenerate critical point of a 2D-vector field  $V$ , the curvature or the perpendicular curvature (or both curvatures) tends to infinity.

4) Many applications of vector field visualization deal with *linear* (or bilinear) *vector fields*. The curvatures of these special vector fields have further characteristic properties:

For linear vector fields (2D or 3D), the curvature along a ray with its origin in the critical point is inversely proportional to the distance to the critical point. The same statement is true for the torsion of tangent curves in linear 3D vector fields.

Between a linear 2D vector field  $V$  and its curvature and perpendicular curvature (both together considered as a new vector field) there is a dual correlation.

5) The visualization of their curvatures is a useful method for *visualizing vector fields*. In the resulting pictures the critical points of the vector fields

are clearly detectable as highlights. Since numerical integrating of the curves is not necessary (in contrast to previous methods of tangent curve visualization), there is no risk of destroying the topology of the original vector field. The thus obtained visualizations of the curvature of vector fields are non-confusing and without overloading or ambiguities.

6) The concept of curvature of tangent curves can be extended to tangent curves on surfaces. They can be described in two ways: as a 3D vector field over the surface and as a 2D vector field in the domain of the surface. Both descriptions can be transferred into each other. Both descriptions give the curvature and the geodesic curvature of the tangent curves on the surface.

7) Visualizing particular tangent curves on surfaces, they appear with a varying "thickness" at different locations on the surface. The "thickness" can be computed and visualized. It reflects certain characteristics of the surface.

8) For *particular tangent curves on surfaces*, the formulas for their curvature, geodesic curvature and "thickness" are shown. Those tangent curves are contour lines, lines of curvature, asymptotic lines, isophotes and reflection lines. Conditions for the appearance of critical points are formulated.

9) The visualization of the tangent curves on surfaces is a useful *surface interrogation method*.

10) The  $G^2$  continuity of lines of curvature and asymptotic lines gives *geometric conditions (necessary and sufficient) for  $G^3$  continuity of surfaces*.

# Holger Theisel: Vector Field Curvature and Applications

Dissertation, eingereicht im November 1995

## Thesen

1) *Tangentenkurven* (stream lines, characteristic lines) sind ein wirkungsvolles Mittel zur Beschreibung, Analyse und Visualisierung von Vektorfeldern.

2) Obwohl Tangentenkurven im allgemeinen nicht als parametrische Kurven beschreibbar sind, können ihre Krümmungen für jeden Punkt des Vektorfeldes (mit Ausnahme kritischer Punkte) bestimmt werden. Die *Krümmung eines Vektorfeldes* kann somit definiert werden als skalares Feld, das in jedem Punkt (mit Ausnahme kritischer Punkte) die Krümmung der Tangentenkurve durch diesen Punkt hat. Die Krümmung von 2D-Vektorfeldern kann auf einfache Weise mit den Konzepten "normiertes Vektorfeld" und "Vektorfeld-Divergenz" beschrieben werden.

3) Für Krümmungen von Vektorfeldern werden folgende *Eigenschaften* gezeigt:

Das rotierte Vektorfeld eines 2D-Vektorfeldes  $V$  entsteht durch Rotieren aller Vektoren von  $V$  um einen festgelegten Winkel. Aus der Krümmung eines 2D-Vektorfeldes und der Krümmung des dazu senkrechten (d.h. mit dem Winkel  $\pi/2$  rotierten) Vektorfeldes ergibt sich die Krümmung aller rotierten Vektorfelder von  $V$ . Weiterhin sind aus der Krümmung und der senkrechten Krümmung von  $V$  die Richtungen aller Vektoren aus  $V$  eindeutig bestimmt. In der Umgebung nichtausgearteter kritischer Punkte eines 2D-Vektorfeldes  $V$  divergiert die Krümmung von  $V$  oder die senkrechte Krümmung von  $V$  (oder beide Krümmungen) gegen Unendlich.

4) Viele praktische Anwendungen der Vektorfeld-Visualisierung arbeiten mit *linearen* (oder bilinearen) *Vektorfeldern*. Für diese speziellen Vektorfelder ergeben sich folgende weitere Eigenschaften:

Für lineare Vektorfelder (2D und 3D) verhält sich die Krümmung entlang eines im kritischen Punkt beginnenden Strahls umgekehrt proportional zum Abstand zum kritischen Punkt. Die gleiche Aussage gilt für die Torsion der Tangentenkurven bei 3D-Vektorfeldern.

Zwischen einem linearen 2D-Vektorfeld und seinem Krümmungs- und senkrechten Krümmungsfeld (beide zusammen als neues Vektorfeld aufgefasst) ergibt sich ein dualer Zusammenhang.

5) Die Visualisierung der Krümmungen ist ein wirkungsvolles Mittel zur *Visualisierung von Vektorfeldern*. In den entstehenden Bildern sind die kritischen Punkte gut als Highlights erkennbar. Da (im Gegensatz zu bisherigen Verfahren der Tangentenkurven-Visualisierung) ein numerisches Integrieren der Kurve entfällt, besteht auch keine Gefahr, dabei die Topologie des originalen Vektorfeldes zu zerstören. Die entstehenden Visualisierungen für Krümmungen von Vektorfeldern sind übersichtlich und nicht mehrdeutig.

6) Das Konzept der Krümmungen von Tangentenkurven kann auf *Tangentenkurven auf Freiformflächen* erweitert werden. Diese sind auf zwei Arten beschreibbar: als 3D-Vektorfeld über der Fläche oder als 2D-Vektorfeld im Parameterraum der Fläche. Beide Beschreibungen können in die jeweils andere überführt werden. Aus beiden Beschreibungen lassen sich Krümmung und geodätische Krümmung der Tangentenkurven auf der Fläche bestimmen.

7) Bei der Visualisierung von bestimmten Tangentenkurven auf Flächen haben diese verschiedene "Dicken" an unterschiedlichen Stellen der Fläche. Diese "Dicke" der Tangentenkurven kann berechnet und visualisiert werden und hängt von Charakteristika der Fläche ab.

8) Für *bestimmte Tangentenkurven auf Flächen* werden die Formeln für Krümmung, geodätische Krümmung und "Dicke" hergeleitet. Diese Tangentenkurven sind Konturlinien, Linien maximaler Krümmung, Asymptotenlinien, Isophoten und Reflexionslinien. Gleichzeitig werden Bedingungen für das Auftreten kritischer Punkte bei diesen Tangentenkurven formuliert.

9) Die Krümmungen dieser Tangentenkurven kann zur Qualitätsanalyse von Flächen (*surface interrogation*) genutzt werden.

10) Aus der  $G^2$ -Stetigkeit von Linien maximaler Krümmung und von Asymptotenlinien werden *geometrische Bedingungen (notwendig und hinreichend) für die  $G^3$ -Stetigkeit von Flächen* hergeleitet.

## **Erklärung**

Hiermit erkläre ich, dass die eingereichte Dissertation von mir selbstständig und ohne fremde Hilfe verfasst wurde. Andere als die angegebenen Hilfsmittel und Quellen habe ich nicht benutzt. Die den benutzten Werken wörtlich oder inhaltlich entnommenen Stellen sind als solche kenntlich gemacht.

Weiterhin erkläre ich, dass ich mich nicht zuvor an der Universität Rostock und auch nicht an einer anderen Universität um den Doktorgrad beworben habe.

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Holger Theisel

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