Cores of Swirling Particle Motion in Unsteady Flows

Tino Weinkauf, Jan Sahner, Holger Theisel, and Hans-Christian Hege, *Member, IEEE*

Abstract—In nature and in flow experiments particles form patterns of swirling motion in certain locations. Existing approaches identify these structures by considering the behavior of stream lines. However, in unsteady flows particle motion is described by path lines which generally gives different swirling patterns than stream lines. We introduce a novel mathematical characterization of swirling motion cores in unsteady flows by generalizing the approach of Sujudi/Haimes to path lines. The cores of swirling particle motion are lines sweeping over time, i.e., surfaces in the space-time domain. They occur at locations where three derived 4D vectors become coplanar. To extract them, we show how to re-formulate the problem using the Parallel Vectors operator. We apply our method to a number of unsteady flow fields.

Index Terms—unsteady flow visualization, feature extraction, particle motion.

1 Introduction

Flow fields play a vital role in many areas. Examples are burning chambers, turbomachinery and aircraft design in industry as well as blood flow in medicine. As the resolution of numerical simulations and experimental measurements like PIV have evolved significantly in the last years, the challenge of understanding the intricate flow structures within massive result data sets has made automatic feature extraction necessary.

Among the features of interest, vortices are the most prominent: they play a major role due to their wanted or unwanted effects on the flow. As an example, Figure 1 shows the vortex in the wake of an airplane. Its influence on the flow at the runway lasts for several minutes and is high enough to cause serious trouble for other airplanes that follow too closely. Understanding vortex structures is important in order to manipulate and control them successfully.

Several definitions of a vortex have been proposed [7, 8, 12, 5] and a number of extraction schemes are based upon them [1, 10, 13, 14]. Thorough overviews of algorithms for the treatment of vortical structures can be found in the literature [11, 10].

One way to assess vortices in experiments is to emit particles into the flow and to examine their behavior: patterns of swirling flow indicate vortices. This has been done in the experiment shown in Figure 1. By injecting smoke, i.e., a huge amount of particles, swirling flow caused by the wake vortex becomes visible. For numerical and measured data sets, Sujudi and Haimes [16] proposed a scheme to extract centers of swirling flow. Peikert et al. formulated the idea of Sujudi/Haimes using the Parallel Vectors operator and presented a fast and robust extraction technique [10]. Bauer et al. [2] and Theisel et al. [17] proposed different algorithms to track these centers over time in unsteady flows. All these approaches have in common that they assess the behavior of *stream lines* only.

However, most flow phenomena are unsteady in nature. In unsteady flows (as shown in Figure 1), particle motion is described by *path lines* instead of stream lines. This generally gives different swirling patterns. In this paper we aim at extracting the cores of swirling particle motion in unsteady flows based on the behavior of path lines. To do so, we develop a novel mathematical characterization of such cores as a generalization of the original idea of Sujudi/Haimes. We do this for 2D and 3D flows. In the latter case, the resulting core structures are lines sweeping over time, i.e., surfaces in the space-time domain.

- *Tino Weinkauf, Jan Sahner, and Hans-Christian Hege are with Zuse Institute Berlin, E-mail: {weinkauf, sahner, hege}@zib.de.*
- *Holger Theisel is with Bielefeld University, E-mail: theisel@techfak.uni-bielefeld.de.*

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Fig. 1. Wake vortex study from NASA Langley Research Center [9]. The flow around a starting agricultural plane is made visible using smoke injection. A huge pattern of swirling particle motion is created by the aircraft's wake vortex. This paper aims at extracting the cores of such areas.

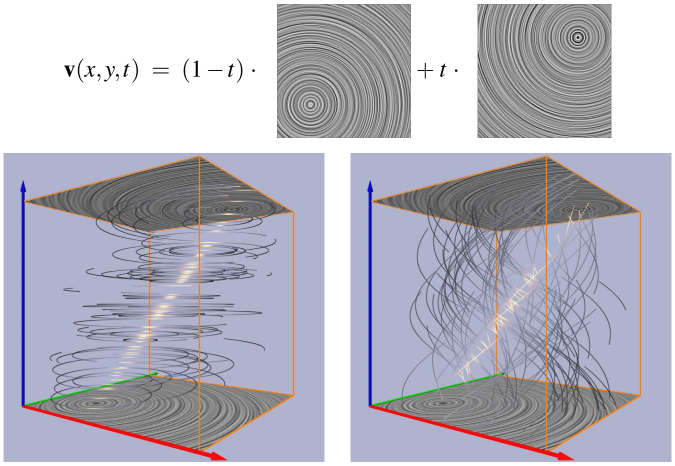
At a single time step, particles group around these core lines forming patterns of swirling motion similar to Figure 1. That is why we refer to those features as *swirling particle cores*. Mathematically, they are characterized by the coplanarity of three 4D vectors. In order to extract them, we show how to re-formulate the problem using the Parallel Vectors operator [10] and apply it accordingly.

The paper is organized as follows: In section 2 we review the basic principles of characteristic curves and swirling motion of stream lines. In section 3 we develop our description of cores of swirling particle motion. Furthermore, we introduce a unified notation of swirling motion in 2D and 3D flows. In section 4 we re-formulate this description using the Parallel Vectors operator and show how to extract the cores. In section 5 we apply our technique to a number of 2D and 3D flows. We draw conclusions in section 6.

2 Theoretical Background

2.1 Stream Lines and Path Lines

In a time-dependent vector field $\mathbf{v}(\mathbf{x},t)$ there are two important types of characteristic curves: stream lines and path lines. In a space-time point $(\mathbf{x}_0,t_0)$ we can start a *stream line* (staying in time slice $t=t_0$) by



(a) Stream lines of $\mathbf{s}$ correspond to the stream lines in $\mathbf{v}$. See (5).

(b) Stream lines of $\mathbf{p}$ correspond to the path lines in $\mathbf{v}$. See (4).

Fig. 2. Characteristic curves of a simple 2D time-dependent vector field shown as illuminated field lines. The red/green coordinate axes denote the (x, y)-domain, the blue axis shows time.

integrating

$$\frac{d}{d\tau}\mathbf{x}(\tau) = \mathbf{v}(\mathbf{x}(\tau), t_0) \quad \text{with} \quad \mathbf{x}(0) = \mathbf{x}_0 \tag{1}$$

or a *path line* by integrating

$$\frac{d}{dt}\mathbf{x}(t) = \mathbf{v}(\mathbf{x}(t), t) \quad \text{with} \quad \mathbf{x}(t_0) = \mathbf{x}_0. \tag{2}$$

Path lines describe the trajectories of massless particles in time-dependent vector fields. The ODE system (2) can be rewritten as an autonomous system at the expense of an increase in dimension by one, if time is included as an explicit state variable:

$$\frac{d}{dt}\begin{pmatrix}\mathbf{x}\\ t\end{pmatrix} = \begin{pmatrix}\mathbf{v}(\mathbf{x}(t),t)\\ 1\end{pmatrix} \quad \text{with} \quad \begin{pmatrix}\mathbf{x}\\ t\end{pmatrix}(0) = \begin{pmatrix}\mathbf{x}_0\\ t_0\end{pmatrix}. \tag{3}$$

In this formulation space and time are dealt with on equal footing – facilitating the analysis of spatio-temporal features. Path lines of the original vector field $\mathbf{v}$ in ordinary space now appear as stream lines of the vector field

$$\mathbf{p}(\mathbf{x}, t) = \begin{pmatrix}\mathbf{v}(\mathbf{x}, t)\\ 1\end{pmatrix} \tag{4}$$

in space-time. To treat stream lines of $\mathbf{v}$, one may simply use

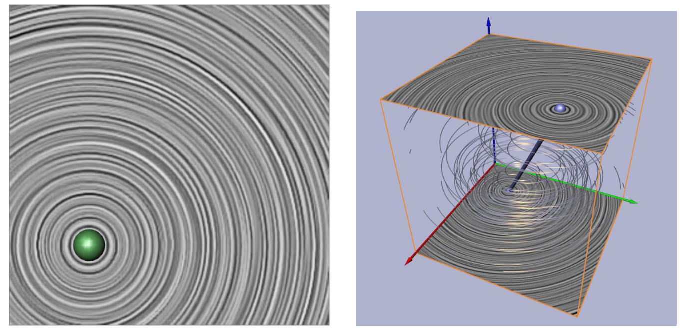
$$\mathbf{s}(\mathbf{x}, t) = \begin{pmatrix}\mathbf{v}(\mathbf{x}, t)\\ 0\end{pmatrix}. \tag{5}$$

Figure 2 illustrates $\mathbf{s}$ and $\mathbf{p}$ for a simple example vector field $\mathbf{v}$. It is obtained by a linear interpolation over time of two bilinear vector fields.

A vector field is called *steady* if $\mathbf{v}(\mathbf{x}, t) = \mathbf{v}(\mathbf{x}, t_0)$, i.e., it is constant over time. In this setting, stream lines and path lines coincide and are given as the solution of

$$\frac{d}{d\tau}\mathbf{x}(\tau) = \mathbf{v}(\mathbf{x}(\tau)) \quad \text{with} \quad \mathbf{x}(0) = \mathbf{x}_0. \tag{6}$$

A number of visualization techniques originally designed for steady vector fields can be applied to unsteady fields by considering each time step independently. In this case equations (6) and (1) coincide – hence, these approaches address the behavior of stream lines. Examining the behavior of path lines (particles) requires to consider time explicitly.



(a) Cores of swirling motion in 2D steady flow fields are critical points where the Jacobian has a pair of conjugate complex eigenvalues.

(b) Cores of spiraling stream lines in 2D unsteady flows are certain critical points (foci and center) tracked over time. Example vector field from Figure 2a.

Fig. 3. Swirling motion of stream lines in 2D steady and unsteady flows.

2.2 Swirling Motion of Stream Lines

Patterns of spiraling stream lines in 2D and 3D flows have already been treated in the literature. These patterns are assessed by examining eigenvalues and eigenvectors of the first derivative $\mathbf{J}$ (the Jacobian) of the respective flow field $\mathbf{v}$. A necessary condition for spiraling stream lines in $\mathbf{v}$ is that $\mathbf{J}$ has a pair of conjugate complex eigenvalues. In the following we give a short overview of the literature.

2.2.1 2D Flows

A steady 2D flow field is given as

$$\mathbf{v}(x, y) = \begin{pmatrix}u(x,y)\\ v(x,y)\end{pmatrix}. \tag{7}$$

The Jacobian of this field has either two real or one pair of conjugate complex eigenvalues. Swirling motion occurs in the latter case only – stream lines spiraling around a common point (see Figure 3a). The velocity at this point must be zero, i.e., $\mathbf{v}(x, y) = \mathbf{0}$. This means that cores of swirling motion in 2D steady flow fields are certain types of critical points, namely foci and centers. Thus, they can be treated using steady flow field topology as described in [6].

An unsteady 2D flow field is given as

$$\mathbf{v}(x, y, t) = \begin{pmatrix}u(x,y,t)\\ v(x,y,t)\end{pmatrix}. \tag{8}$$

Following (5), the stream lines of this field always stay in the same given time slice t_0 (Figure 2a). Thus, swirling motion in a single time slice can be captured by applying the scheme known from the steady case. By changing the given time slice, the critical points will move over time and form line-type structures in space-time. In order to extract all locations of all foci and centers one has to apply algorithms for tracking critical points as known from [20, 18, 19]. Figure 3b shows this for the simple example vector field known from Figure 2a.

Throughout the paper, swirling stream line cores will be colored blue, whereas swirling particle cores will be colored red.

2.2.2 3D Flows

A steady 3D flow field is given as

$$\mathbf{v}(x, y, z) = \begin{pmatrix}u(x,y,z)\\ v(x,y,z)\\ w(x,y,z)\end{pmatrix}. \tag{9}$$

Centers of swirling motion in these fields have first been treated by Sujudi and Haimes [16]. Their inspiration was the flow pattern around a certain type of critical point: a focus saddle (see Figure 4 for an illustration). Here, the Jacobian of the flow field has one real and two

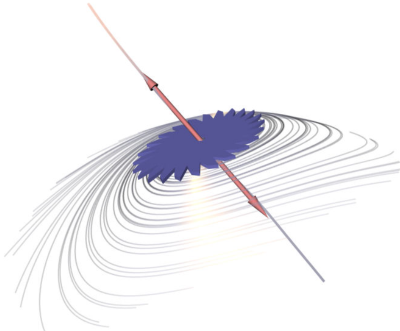
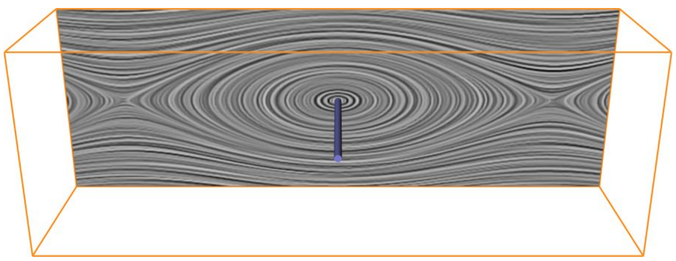
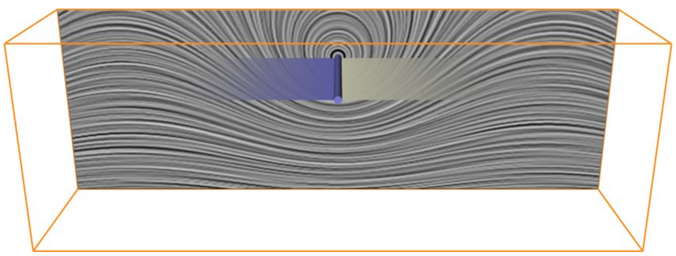


Fig. 4. Flow pattern around a focus saddle. The blue plane is spanned by the eigenvectors corresponding to the complex eigenvalues. Swirling motion takes place in this plane around the real eigenvector denoted by the red arrows.



(a) Cores of swirling motion in 3D steady flows are lines where $\mathbf{v}||\mathbf{e}$. Shown is a simple vortex model: the Stuart vortex (11).



(b) Cores of spiraling stream lines in 3D unsteady flows are lines sweeping over time. Blue denotes the past of the core, gray shows the future. Shown is a certain time step of the Stuart vortex moving over time from left to right with a constant velocity. This added convection velocity results in an upward shift of the stream line core compared to the steady case. The LIC plane shows the stream line pattern at the chosen time step.

Fig. 5. Swirling motion of stream lines in 3D steady and unsteady flows.

complex eigenvalues. The eigenvectors corresponding to the complex eigenvalues span a plane in which the flow spirals around the critical point. The eigenvector corresponding to the real eigenvalue denotes the axis of rotation.

Sujudi and Haimes generalized this flow pattern to non-critical points by considering the so called reduced velocity. At a point $\mathbf{x}$, the reduced velocity $\mathbf{w}(\mathbf{x})$ is given as the projection of the steady flow field $\mathbf{v}(\mathbf{x})$ to the plane normal to the real eigenvector $\mathbf{e}$ by

$$\mathbf{w}(\mathbf{x}) = \mathbf{v}(\mathbf{x}) - (\mathbf{v}(\mathbf{x}) \cdot \mathbf{e}(\mathbf{x}))\,\mathbf{e}(\mathbf{x}). \tag{10}$$

They show that centers of swirling flow are line-type structures where $\mathbf{w}(\mathbf{x}) = 0$. Peikert et al. [10] formulated this using the Parallel Vectors operator and showed that $\mathbf{w}(\mathbf{x}) = 0$ is equivalent to $\mathbf{v}(\mathbf{x})||\mathbf{e}(\mathbf{x})$, i.e., $\mathbf{v}$ and $\mathbf{e}$ are parallel. Figure 5a shows the core line of swirling flow for the Stuart vortex, a simple vortex model given by

$$\mathbf{v}(x,y,z) = \begin{pmatrix} \sinh(y)/(\cosh(y) - 0.25\cos(x)) \\ -0.25\sin(x)/(\cosh(y) - 0.25\cos(x)) \\ z \end{pmatrix}. \tag{11}$$

As expected, the swirling patterns in the LIC image correlate with the extracted center of swirling flow.

An unsteady 3D flow field is given as

$$\mathbf{v}(x,y,z,t) = \begin{pmatrix} u(x,y,z,t) \\ v(x,y,z,t) \\ w(x,y,z,t) \end{pmatrix}. \tag{12}$$

At each time step t_0, locations of swirling stream line behavior can be extracted using the method known from the steady case. This yields lines sweeping over time, resulting in 4D surfaces in the space-time domain. Two approaches for extracting those surfaces exist [2, 17]. Both address swirling motion of stream lines, not particles.

Figure 5b gives an example for sweeping stream line cores, extracted by the method in [17]. The flow field is derived from the steady model (11) by superimposing a constant flow of velocity $(1,0,0)^T$. This leads to structures moving constantly from left to right over time. The swirling stream line core at $t = 0$ is depicted as a blue line. Past and future is encoded in the surface, where blue means the past and grey encodes future. Note that by adding the convection velocity, the swirling stream line loci moved up relative to the steady case.

3 Cores of Swirling Particle Motion

In the last section we reviewed the approaches for extracting swirling motion of *stream lines* in steady and unsteady flows. To the best of our knowledge, there exists no approach for capturing swirling motion of *path lines*. We believe that studying the behavior of path lines is important since particle motion in unsteady flows is described by path lines instead of stream lines. Furthermore, many flow phenomena are unsteady and examining them solely based on the behavior of stream lines in certain time steps may not give the complete picture.

In the course of this section we develop a novel mathematical characterization of swirling particle motion in 2D and 3D unsteady flows – sections 3.1 and 3.2 respectively. Afterwards, we give a comprehensive summary of all discussed types of swirling flow leading to a more generalized notation (section 3.3).

3.1 Swirling Particle Motion in 2D

Following (4), the path lines of an unsteady 2D flow field $\mathbf{v}(x,y,t)$ are given as the stream lines of the steady 3D vector field

$$\mathbf{p}(x,y,t) = \begin{pmatrix} \mathbf{v}(x,y,t) \\ 1 \end{pmatrix} = \begin{pmatrix} u(x,y,t) \\ v(x,y,t) \\ 1 \end{pmatrix}. \tag{13}$$

This formulation of path lines as stream lines in a higher-dimensional vector field reduces the identification of swirling particle motion to a known case: it can be treated similar to swirling stream line motion of a steady 3D vector field by applying the original approach of Sujudi/Haimes [16] to $\mathbf{p}$. This yields line structures where the Jacobian of $\mathbf{p}$ has a pair of conjugate complex eigenvalues and the only real eigenvector is parallel to $\mathbf{p}$. The Jacobian of $\mathbf{p}$ is

$$\mathbf{J}(\mathbf{p}) = \begin{bmatrix} u_x & u_y & u_t \\ v_x & v_y & v_t \\ 0 & 0 & 0 \end{bmatrix} \tag{14}$$

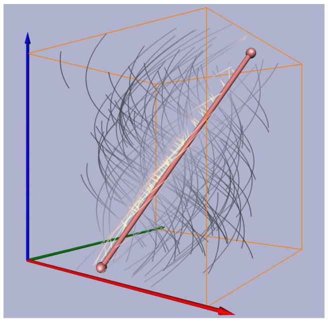
and has the eigenvalues $e_1, e_2, 0$ with the respective eigenvectors

$$\begin{pmatrix} \mathbf{e}_1 \\ 0 \end{pmatrix}, \begin{pmatrix} \mathbf{e}_2 \\ 0 \end{pmatrix}, \mathbf{f}, \tag{15}$$

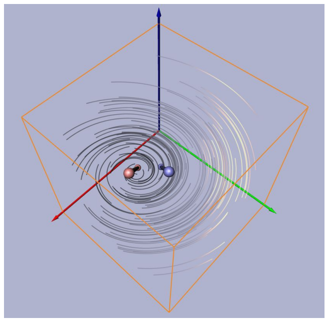
where $e_1, e_2, \mathbf{e}_1, \mathbf{e}_2$ constitute the eigensystem of the spatial Jacobian $\left[\begin{smallmatrix} u_x & u_y \\ v_x & v_y \end{smallmatrix}\right]$ and

$$\mathbf{f} = \begin{pmatrix} \det(\mathbf{v}_y, \mathbf{v}_t) \\ \det(\mathbf{v}_t, \mathbf{v}_x) \\ \det(\mathbf{v}_x, \mathbf{v}_y) \end{pmatrix}. \tag{16}$$

Note, that $\mathbf{f}$ is the so-called Feature Flow Field for tracking critical points in time-dependent 2D vector fields known from [18]. In order to track critical points of $\mathbf{v}$, this field $\mathbf{f}$ was designed such that the



(a) View similar to Figure 2b.



(b) View from top. Additionally, the core line of spiraling stream lines is shown in blue (cf. Figure 3b).

Fig. 6. Core line (shown in red) of swirling particle motion of a simple 2D unsteady vector field $\mathbf{v}$ (see Figure 2). The path lines of $\mathbf{v}$ spiral around this core line.

values of $\mathbf{v}$ do not change along the stream lines of $\mathbf{f}$. In other words, the directional derivative of $\mathbf{v}$ in direction of $\mathbf{f}$ is zero. This means that $\mathbf{J}(\mathbf{v}) \cdot \mathbf{f} = \mathbf{0}$ and consequently $\mathbf{J}(\mathbf{p}) \cdot \mathbf{f} = 0 \cdot \mathbf{f}$. Hence, $\mathbf{f}$ necessarily is an eigenvector of $\mathbf{J}(\mathbf{p})$ corresponding to the eigenvalue 0.

Figure 6 shows the core of swirling particle motion for the simple 2D unsteady vector field introduced in Figure 2. The core line is depicted in red. Additionally, Figure 6b shows the core line of spiraling stream lines in blue. It can clearly be seen that both structures are different and that the particle core line lies in the center of the spiraling path lines.

3.2 Swirling Particle Motion in 3D

We aim at extracting swirling particle cores of an unsteady 3D flow field $\mathbf{v}(x,y,z,t)$, i.e., locations around which spiraling patterns of path lines occur. Following (4), path lines of $\mathbf{v}$ are stream lines of the steady 4D vector field

$$\mathbf{p}(x,y,z,t) = \begin{pmatrix} \mathbf{v}(x,y,z,t) \\ 1 \end{pmatrix} = \begin{pmatrix} u(x,y,z,t) \\ v(x,y,z,t) \\ w(x,y,z,t) \\ 1 \end{pmatrix}. \tag{17}$$

There is no existing tool to identify swirling motion in a steady 4D vector field. In the following we develop a new approach. The key is, again, the eigensystem of the Jacobian of $\mathbf{p}$. In the 3D unsteady setting, the Jacobian of $\mathbf{p}$ is

$$\mathbf{J}(\mathbf{p}) = \begin{bmatrix} u_x & u_y & u_z & u_t \\ v_x & v_y & v_z & v_t \\ w_x & w_y & w_z & w_t \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{18}$$

and has the eigenvalues $e_1, e_2, e_3, 0$ with the respective four eigenvectors

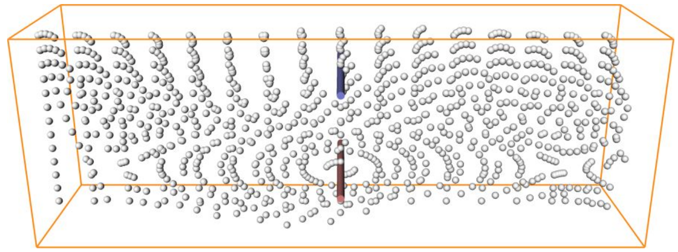
$$\begin{pmatrix} \mathbf{e}_1 \\ 0 \end{pmatrix}, \begin{pmatrix} \mathbf{e}_2 \\ 0 \end{pmatrix}, \begin{pmatrix} \mathbf{e}_3 \\ 0 \end{pmatrix} =: \mathbf{e}^s, \mathbf{f}, \tag{19}$$

where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are the eigenvectors of the spatial Jacobian

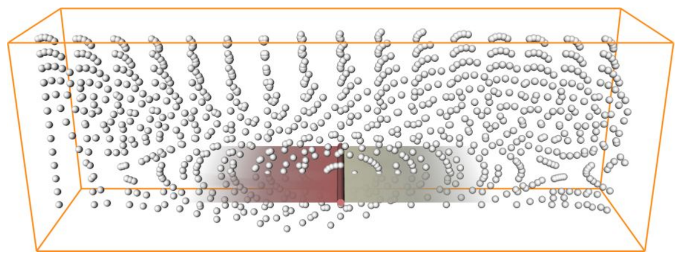
$$\mathbf{J}_s(\mathbf{v}) = \begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{bmatrix} \tag{20}$$

and the fourth eigenvector $\mathbf{f}$ can be written as

$$\mathbf{f}(x,y,z,t) = \begin{pmatrix} +\det(\mathbf{v}_y, \mathbf{v}_z, \mathbf{v}_t) \\ -\det(\mathbf{v}_z, \mathbf{v}_t, \mathbf{v}_x) \\ +\det(\mathbf{v}_t, \mathbf{v}_x, \mathbf{v}_y) \\ -\det(\mathbf{v}_x, \mathbf{v}_y, \mathbf{v}_z) \end{pmatrix}. \tag{21}$$



(a) Swirling particle core as a line at $t = 0$ shown in red. Particles are grouping around that line in a spiraling pattern. Additionally, the core of swirling stream line motion is shown in blue. This clearly shows that stream lines and path lines spiral around different centers (compare with Figure 5b).



(b) Swirling particle cores in 3D unsteady flows are lines sweeping over time, i.e., surfaces in the 4D space-time domain. Red denotes the past of the core line, gray shows its future.

Fig. 7. As an illustration of our new technique, swirling particle motion is shown in the example of the Stuart vortex moving over time from left to right with a constant velocity.

Note, that $\mathbf{f}$ is the Feature Flow Field for tracking critical points in 3D unsteady vector fields [21] – here, the same explanation as in the 2D case applies (section 3.1). The eigenvalue corresponding to $\mathbf{f}$ is always zero. Therefore, $\mathbf{J}(\mathbf{p})$ has always one real eigenvalue and only the following cases can occur:

- All eigenvalues of $\mathbf{J}(\mathbf{p})$ are real.
- $\mathbf{J}(\mathbf{p})$ has a pair of conjugate complex eigenvalues and two real eigenvalues – let them be sorted such that e_1, e_2 are complex and e_3 is real.

Since complex eigenvalues are a necessary condition, swirling motion is only possible in the latter case. At any given point $\mathbf{x}$ in the 4D domain, the eigenvectors corresponding to e_1, e_2 span a plane P_c in which locally the swirling motion occurs. The two real eigenvectors $\mathbf{e}^s$ and $\mathbf{f}$ denote the part of the flow which is independent of swirling – they span a plane P_r in which no swirling occurs at all. In order to see what the core of swirling motion in 4D is, consider the following rephrasing of the definitions of swirling motion cores in other dimensions:

> *Although a point* $\mathbf{x}$ *on the core structure is surrounded by spiraling integral curves, the flow vector at* $\mathbf{x}$ *itself is solely governed by the non-swirling part of the flow.*

This is a direct generalization of the "reduced velocity"-idea of Sujudi/Haimes. For our case this means that $\mathbf{x}$ is a point on the swirling particle core if the flow vector $\mathbf{p}(\mathbf{x})$ lies in the plane of non-swirling flow P_r, i.e., the plane spanned by $\mathbf{e}^s$ and $\mathbf{f}$. In other words, the swirling particle cores are at locations where

$$\lambda_1 \mathbf{p} + \lambda_2 \mathbf{e}^s + \lambda_3 \mathbf{f} = \mathbf{0} \quad \text{with} \quad \lambda_1^2 + \lambda_2^2 + \lambda_3^2 > 0. \tag{22}$$

This is a coplanarity problem: swirling particle cores are at locations where the 4D vectors $\mathbf{p}$, $\mathbf{e}^s$ and $\mathbf{f}$ are coplanar. We call the operator solving this equation the *Coplanar Vectors* operator, which reads in the general setting

$$\lambda_1 \mathbf{a} + \lambda_2 \mathbf{b} + \lambda_3 \mathbf{c} = \mathbf{0} \quad \text{with} \quad \lambda_1^2 + \lambda_2^2 + \lambda_3^2 > 0. \tag{23}$$

In order to show that our approach is reasonable, we study what happens if we require the flow field $\mathbf{v}$ to be steady, i.e., $\mathbf{v}(x,y,z,t) = \mathbf{v}(x,y,z,t_0)$. In this setting, path lines coincide with stream lines and our approach needs to reduce to the steady case, i.e., the method of Sujudi/Haimes. As the temporal derivative $\mathbf{v}_t = 0$, the fourth eigenvector becomes $\mathbf{f} = (0,0,0,-\det(\mathbf{v}_x,\mathbf{v}_y,\mathbf{v}_z))^T$ following (21), and the coplanarity condition (22) reads

$$\lambda_1 \begin{pmatrix} \mathbf{v} \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} \mathbf{e}_3 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\det(\mathbf{v}_x,\mathbf{v}_y,\mathbf{v}_z) \end{pmatrix} = \mathbf{0}. \qquad (24)$$

The last component of this equation requires $\lambda_1 = \lambda_3 \det(\mathbf{v}_x,\mathbf{v}_y,\mathbf{v}_z)$. Hence, in the steady setting our approach reduces to $\mathbf{v}||\mathbf{e}_3$, i.e., the method of Sujudi/Haimes.

In Figure 7 the core of swirling particle motion has been extracted and visualized for the moving Stuart vortex. In Figure 7a the core is shown for $t = 0$ as a red line. A number of particles have been seeded uniformly and advected over time. They form a spiraling pattern around the core line. That is why we call the structures fulfilling (22) *swirling particle cores*. Additionally, Figure 7a shows the core of swirling stream line motion in blue. It can clearly be seen that the structures are different. Note, that swirling particle cores of 3D unsteady fields are surfaces, i.e., the particle core lines of a single time step sweep over time. This is shown in Figure 7b where the past of the core line is encoded in red and its future in gray.

In section 4 we will show how to extract swirling particle cores. In the next section we will give a summary on swirling motion in general.

3.3 A Unified Notation of Swirling Motion Cores

As we have seen in the previous sections, one may find swirling motion of

- stream lines in steady fields,
- stream lines in unsteady fields,
- path lines in unsteady fields.

All three cases can be found in 2D as well as 3D fields, summing up to a total of six cases. In the following we show how all those cases can be written using a unified notation.

Let $\mathbf{V}$ be the autonomous system of the characteristic curves in question, i.e., $\mathbf{v}$ for the steady case (6), $\mathbf{s}$ for the stream lines of an unsteady flow (5), and $\mathbf{p}$ for path lines (4). Let $\mathbf{e}_i$ be the eigenvectors corresponding to the real eigenvalues of $\mathbf{J}(\mathbf{V})$. The point $\mathbf{x}$ is part of the respective core of swirling motion, if $\mathbf{V}(\mathbf{x})$ lies in the span of $\mathbf{e}_i(\mathbf{x})$. In other words, this reads

$$\lambda_1 \mathbf{V}(\mathbf{x}) + \sum \lambda_i \mathbf{e}_i(\mathbf{x}) = \mathbf{0} \quad \text{with} \quad \sum \lambda_i^2 > 0. \qquad (25)$$

We want to illustrate this: depending on the dimension of $\mathbf{V}$, this is equivalent to

- extraction of critical points in 2D,
- solving the Parallel Vectors operator in 3D,
- solving the Coplanar Vectors operator in 4D.

Table 1 illustrates this. The first column of this table covers the steady case, i.e., critical point extraction in 2D and the method of Sujudi/Haimes in 3D. They have been discussed in section 2.2. The third column refers to swirling particle cores as developed above. It remains to explain the second column where swirling motion of stream lines in unsteady flows is treated. In the 2D unsteady case, the Parallel Vectors operator is applied to $\mathbf{s}$ and its only real eigenvector $\mathbf{e}$. We need to show that this describes critical points (foci and centers) tracked over time as discussed in section 2.2.1. Indeed,

$$\lambda_1 \mathbf{s} + \lambda_2 \mathbf{e} = \lambda_1 \begin{pmatrix} u \\ v \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} = 0 \qquad (26)$$

requires $\lambda_2 = 0$ and hence $\mathbf{v} = \binom{u}{v} = 0$, as needed. This means that the Parallel Vectors operator can be used to track critical points in time-dependent 2D vector fields. In fact, the critical point in Figure 3b has been tracked this way. See also Figures 9a-b.

A similar statement holds for the 3D unsteady case, where the Coplanar Vectors operator is used to describe swirling stream line cores swept over time.

4 Extraction of Swirling Particle Cores

In the following we show how to extract swirling particle cores of unsteady 3D flows. In the next section we re-formulate the coplanarity problem (22) using the Parallel Vectors operator [10] – a common tool for feature extraction in the visualization community, which we will briefly explain in section 4.2.

4.1 Formulation using Parallel Vectors

We identified cores of swirling particle motion in unsteady 3D flows as locations where the three 4D vector fields $\mathbf{p}, \mathbf{e}^s, \mathbf{f}$ are coplanar. This is given by (22), which reads component-wise

$$\lambda_1 \begin{pmatrix} u \\ v \\ w \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} \mathrm{e}_1^s \\ \mathrm{e}_2^s \\ \mathrm{e}_3^s \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} \mathrm{f}_1 \\ \mathrm{f}_2 \\ \mathrm{f}_3 \\ \mathrm{f}_4 \end{pmatrix} = \mathbf{0}, \qquad (27)$$

By setting $\lambda_1 = -\lambda_3 \mathrm{f}_4$ we can eliminate the fourth component, and the reformulation reads

$$\lambda_2 \underbrace{\begin{pmatrix} \mathrm{e}_1^s \\ \mathrm{e}_2^s \\ \mathrm{e}_3^s \end{pmatrix}}_{\mathbf{a}} + \lambda_3 \underbrace{\left(\begin{pmatrix} \mathrm{f}_1 \\ \mathrm{f}_2 \\ \mathrm{f}_3 \end{pmatrix} - \mathrm{f}_4 \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right)}_{\mathbf{b}} = \mathbf{0}. \qquad (28)$$

This is a 3D Parallel Vectors problem. The reformulation $\mathbf{a}||\mathbf{b}$ is equivalent to the coplanarity of the vector fields $\mathbf{p}, \mathbf{e}^s, \mathbf{f}$, and hence $\mathbf{a}||\mathbf{b}$ is satisfied exactly at the cores of swirling particle motion in unsteady flow fields. With this reformulation at hand we can use the powerful extraction techniques available for the Parallel Vectors operator.

Note that although the eigenvectors corresponding to the eigenvalue zero can be calculated explicitly using formulae (16) and (21), it is more stable to calculate all involved eigenvectors using an eigenvector solver, especially in degenerate cases where $\det(\mathbf{v}_x,\mathbf{v}_y) = 0$ in 2D or $\det(\mathbf{v}_x,\mathbf{v}_y,\mathbf{v}_z) = 0$ in 3D.

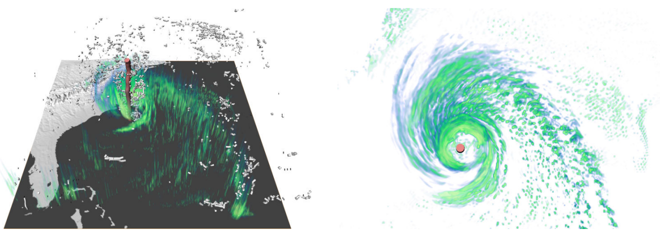
4.2 Extraction using Parallel Vectors

Applying the Parallel Vectors operator to our problem yields lines sweeping over time, i.e., surfaces in the 4D space-time domain. Different methods of extracting these surfaces exist [2, 17] – both being extensions of the original approach presented in [10], which can be summarized as follows: to extract the solution lines for a specific time step, one has to iterate over all faces of the grid and search for points where $\mathbf{a}||\mathbf{b}$. For example, in a triangle (being the face of a tetrahedral mesh) one can find such points by solving a certain three-by-three eigenvalue problem. The extracted points have to be connected to lines in a post-processing step by considering the solution points at the faces belonging to the same volume element (tetrahedron, voxel, etc.). On volume elements with more than two adjacent solutions, a decision is necessary which points should be connected. We use a simple angle criterion: the line segments with the smallest angle to each other are connected.

As Peikert et al. [10] already pointed out, one expects swirling stream line cores to point in direction of the flow field, but the parallelity condition $\mathbf{a}||\mathbf{b}$ does not ensure this. Indeed, the solution lines can be orthogonal to the input vectors. Whenever this is not desired, one can filter the output such that only lines are displayed that do not exceed a defined threshold angle towards $\mathbf{a}$, or $\mathbf{b}$. For both the swirling stream line and particle cores we use the angle between the solution lines and $\mathbf{e}$ as a criterion.

	steady	unsteady	
	stream lines	stream lines	path lines
2D	CRITICAL POINTS $\lambda \mathbf{v} = \mathbf{0}$ *CP finder* can be treated using [6]	TRACKED CRITICAL POINTS $\lambda_1 \mathbf{s} + \lambda_2 \mathbf{e} = \mathbf{0}$ *PV operator* can be treated using [18] and, as proven, using [16, 10]	SWIRLING PARTICLE CORES $\lambda_1 \mathbf{p} + \lambda_2 \mathbf{e} = \mathbf{0}$ *PV operator* treated in this paper can be extracted using [16, 10]
3D	SWIRLING STREAM LINE CORES $\lambda_1 \mathbf{v} + \lambda_2 \mathbf{e} = \mathbf{0}$ *PV operator* original idea of Sujudi/Haimes treated in [16, 10]	TRACKED STREAM LINE CORES $\lambda_1 \mathbf{s} + \lambda_2 \mathbf{e}^s + \lambda_3 \mathbf{f} = \mathbf{0}$ *CV operator* treated in [2, 17]	SWIRLING PARTICLE CORES $\lambda_1 \mathbf{p} + \lambda_2 \mathbf{e}^s + \lambda_3 \mathbf{f} = \mathbf{0}$ *CV operator* treated in this paper

Table 1. Summary of swirling motion in 2D and 3D flows. Depending on the dimension of the autonomous system the conditions can be written using the notations of Critical Points (CP), Parallel Vectors (PV), and Coplanar Vectors (CV).



(a) Complete extraction result prior to filtering. Lines shorter than a threshold are depicted in gray.

(b) Orthographic view from top showing the dominant particle core (red) in the center of the clouds.

Fig. 8. Hurricane Isabel data set at $t = 33.5$. Shown are the dominating swirling particle core line (red) and a volume rendering of the cloud moisture mixing ratio.

Note that the extraction of core lines of swirling motion is nonlinear in general, since eigenvectors do not depend linearly on the input fields. While this makes the extraction using linear techniques more difficult in general, we found that filtering the resulting lines by length was sufficient to rule out the nonlinearity.

5 APPLICATIONS

As we use the Parallel Vectors operator for the extraction of swirling particle cores, we refer the reader to the literature [10] for the discussion of timings and memory consumption. In our implementation, a single time step of dimension 128^3 is processed in a single thread in about 15 seconds on an AMD64 X2 4400+.

In 3D unsteady flows, swirling particle cores are lines sweeping over time, i.e., 4D surfaces. In the previous sections we used semi-transparent surfaces to encode past and future (e.g. Figure 7b). However, for more complex data sets these surfaces might contain self intersections. We found that displaying the core lines only – at a certain time step or in an animation – results in clearer visualizations in most cases.

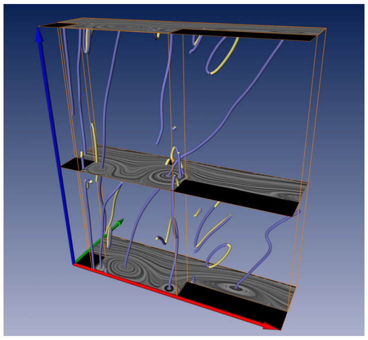
In Figure 8 we applied our method to the Hurricane Isabel data set from the IEEE Visualization 2004 contest. This is a complex 3D time-dependent data set produced by the Weather Research and Forecast (WRF) model. Figure 8a shows the unfiltered extraction result at $t = 33.5$ consisting of 1533 core lines. We have chosen to filter all lines shorter than 10% of the diagonal of the bounding box. The result is the single swirling particle core line in the eye of the hurricane – verified by the volume rendering of the cloud moisture (Figure 8b).

Figure 9 shows an unsteady flow over a 2D cavity. This data set was kindly provided by Mo Samimy and Edgar Caraballo (both Ohio State University) [4] as well as Bernd R. Noack and Ivanka Pelivan (both TU Berlin). 1000 time steps have been simulated using the *compressible* Navier-Stokes equations. The data is almost periodic, with a period of about 100 time steps in length, and only the first 100 time steps are shown.

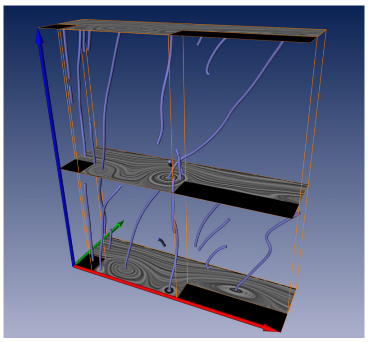
As shown in section 3.3, the Parallel Vectors operator can be used to track certain critical points (foci and centers) over time. Figures 9a-b exemplify this: the blue lines denote swirling motion of stream lines – once extracted by tracking critical points using Feature Flow Fields and once by applying the Parallel Vectors operator. Both results coincide very well. Note that additionally Figure 9a shows tracked saddle points as yellow curves. Figure 9c stresses again the difference between swirling particle and stream line cores: the blue swirling stream line core goes through the center of spiraling stream lines at a specific time step (shown as LIC plane), but it does not lie in the center of spiraling path lines (shown as illuminated lines). Since unsteady motion is described by path lines, existing approaches fail to capture swirling motion in unsteady flows correctly: they are based on stream lines. Our approach captures this behavior correctly as shown by the red swirling particle core. Figure 9d shows the 181 extracted particle core lines, where the majority (154) is shorter than 3.5% of the diagonal of the bounding box and has been filtered accordingly. Figure 9e shows the filtered result.

Figure 10 demonstrates the results of our method applied to a flow behind a circular cylinder. The data set was derived by Bernd R. Noack (TU Berlin) from a direct numerical Navier Stokes simulation by Gerd Mutschke (FZ Rossendorf). It resolves the so called 'mode B' of the 3D cylinder wake at a Reynolds number of 300 and a spanwise wavelength of 1 diameter. The data is provided on a $265 \times 337 \times 65$ curvilinear grid as a low-dimensional Galerkin model. The flow exhibits periodic vortex shedding leading to the well known von Kármán vortex street [22]. This phenomenon plays an important role in many industrial applications, like mixing in heat exchangers or mass flow measurements with vortex counters. However, this vortex shedding can lead to undesirable periodic forces on obstacles, like chimneys, buildings, bridges and submarine towers.

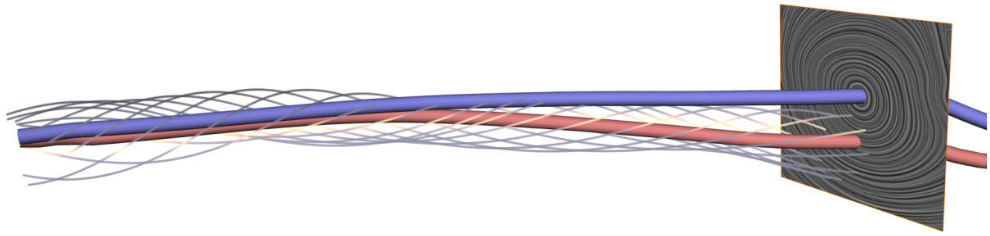
Figures 10a-b show particles seeded at a vertical line on the left-hand side of the bounding box. Due to the periodic vortex shedding these particles form patterns of swirling motion after some integration steps – a clear indication of the von Kármán vortex street. These patterns perfectly match up with the cores of swirling particle motion (red) extracted using our method (filtered by angle criterion with $|\cos(\ell, \mathbf{e})| < 0.3$ and by length with 0.1%). Figure 10c shows stream lines at a certain time step as depicted by the LIC plane: existing approaches based on stream lines have to fail to detect the von Kármán vortex street here, since the original frame of reference does not exhibit any spiraling stream lines. However, our method is based on the behavior of path lines and captures the vortex street correctly. Stream line based approaches are able to capture the features only if one chooses a reference frame matching their convection velocity. In this frame of reference, swirling motion of stream lines is present and



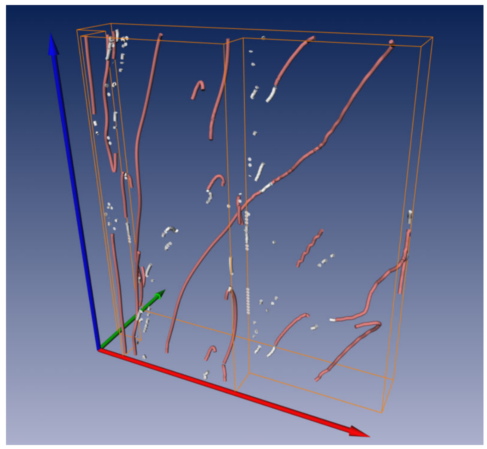
(a) Time evolution of critical points.



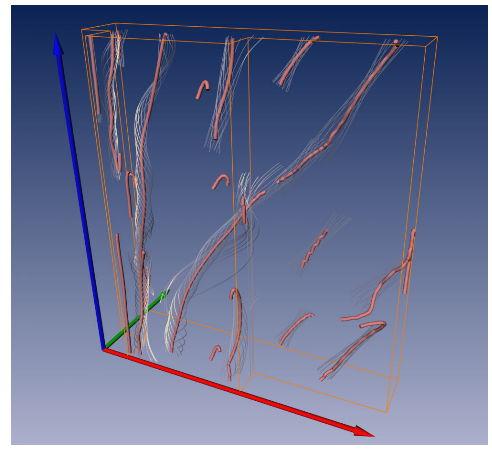
(b) Swirling stream line cores.



(c) Close-up of a swirling particle core (red) in comparison to a swirling stream line core (blue).



(d) Swirling particle cores as identified by our new approach. Short, filtered lines are depicted in gray.



(e) Illuminated path lines verify that our cores are centers of swirling particle motion.

Fig. 9. Unsteady flow over a 2D cavity. Red and green axes span the spatial domain, the blue axis denotes time.

its cores can be extracted by the method of Sujudi/Haimes. This has been done in Figure 10d by applying a priori knowledge [22]. As a reference, the swirling particle cores (red) are also displayed. The extracted structures are very close to each other (Figure 10e). However, our method is able to extract these features in the original frame of reference without a priori knowledge.

6 Conclusions and Discussion

In this paper, we made the following contributions:

- For the first time, we addressed the identification of cores of swirling motion of path lines in unsteady flows.
- We developed a mathematical characterization of swirling particle cores.
- We introduced the Coplanar Vectors operator for deducing the characterization for 3D unsteady flows.
- We showed how to re-formulate and extract swirling particle cores in 3D unsteady fields using the Parallel Vectors operator – a common tool for feature extraction in the visualization community. This eases the implementation of our approach in other visualization systems.
- We presented a unified notation of swirling motion in 2D and 3D flows.
- We applied our technique to a number of 2D and 3D unsteady data sets.

Our method is a generalization of the approach by Sujudi/Haimes and clearly inherits its limitations. In particular, the method can result in false positives, i.e., lines that actually lie off the desired core line. Also when noise is present the method may extract a variety of short lines. Both issues have been treated in the literature [10] by filtering the output by length and certain angle criteria as discussed in section 4.2. Also, Roth and Peikert pointed out in [12] that the method of Sujudi/Haimes has its limitations in settings where curved boundaries are involved. Our method might have comparable limitations and it is an interesting point for further studies to see if the higher-order methods in [12] can be extended to path lines in an analogous way.

A different notion of swirling motion in unsteady flows is currently under development and intermediate results are available [3].

Note that different vortex definitions exist and swirling motion is only one way to assess vortices. While most vortex definitions given in the fluid dynamics community are either Galilean invariant or even objective (see [5, 7, 8, 13, 14]), all types of swirling motion cores are not invariant under such transformations. However, cores of swirling motion have an intuitive interpretation, accompany the behavior of integral curves and their extraction is comparatively simple and fast.

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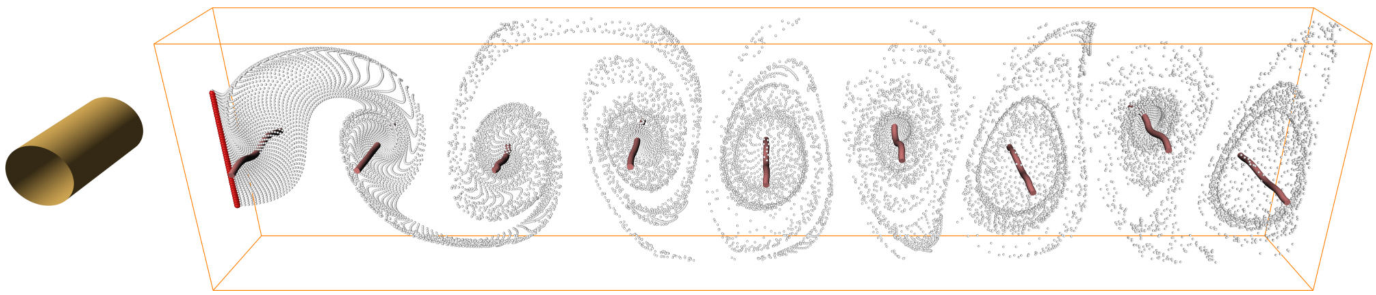
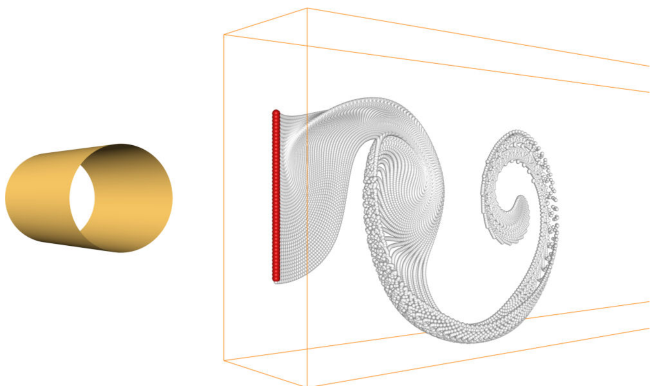
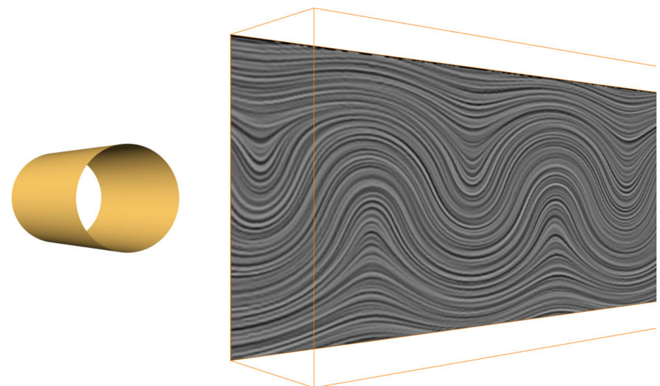


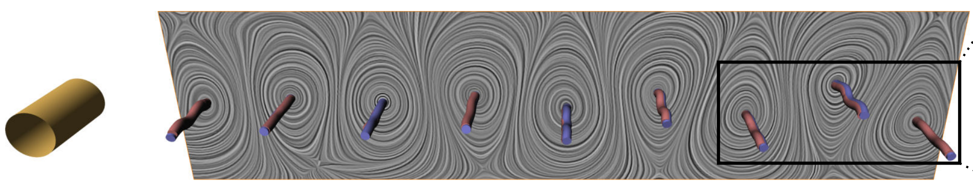
(a) Swirling particle cores (red lines) at a certain time step. The additionally shown particles verify that our core lines are at the centers of swirling particle motion.



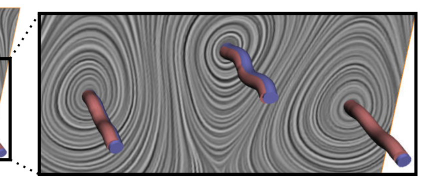
(b) Particles are injected constantly at the vertical line on the left-hand side of the bounding box and advected over time. They form patterns of swirling motion indicating the von Kármán vortex street.



(c) Stream lines at a certain time step visualized using a LIC plane. Swirling motion of stream lines can not be observed in the original frame of reference.



(d) In an *appropriately chosen* frame of reference, swirling motion of stream lines is present (indicated by the LIC plane). The extracted swirling stream line cores (blue) are displayed together with the swirling particle cores (red) of the same time step.



(e) Close-up of (d). Although extracted with different methods in different reference frames, the extracted lines are very similar.

Fig. 10. 3D unsteady flow behind a cylinder. Existing stream line based approaches fail to capture swirling motion cores in the original frame of reference. Our new path line based method is able to extract such features without a priori knowledge.

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